Multi-dimensional Bidding in Two-player Auctions

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We introduce a general model of auctions involving two players. In this model, each player selects a monetary bid and an effort level; a general score function then combines these inputs to determine the winner. This framework unifies standard all-pay contests and winner-pay auctions by allowing a continuous tradeoff between effort and bidding. We prove the existence of a Nash equilibrium under mild technical conditions on the score function. In symmetric contests, we derive explicit expressions for equilibrium bids and efforts, which reveal how the degree of substitutability between effort and bidding influences strategic allocations. We characterize the equilibrium structure in asymmetric settings. Furthermore, we examine how complementarity between effort and bid inputs can amplify or attenuate competitive pressures. Our theoretical contributions include a constructive equilibrium solution in closed form and new insights into multi-dimensional bidding strategies under general score rules.

Keywords: All-pay auction, Bidding strategy, Contest theory, Conditional investment, Score function.

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I. Introduction

Auction theory encompasses a rich variety of formats and information structures, ranging from classical Bayesian auctions with private information to complete-information contests. The present work falls into the latter tradition, and it studies a hybrid contest-auction mechanism under complete information, where each bidder expends two types of effort: an unconditional investment, which is a sunk cost akin to an all-pay auction bid, and a conditional investment, which is a payment made only if they win, as in a standard winner-pay auction. This places our model squarely in the literature on contests and all-pay auctions, particularly the emerging subfield of hybrid contests that mix all-pay and winner-pay elements. In contrast to the well-known Bayesian auction models, such as Myerson (1981), where bidders hold private information and only the winner pays, our setup assumes complete information and requires all players to forfeit some resources regardless of the outcome. This approach is rooted in rent-seeking contest theory, including works by Tullock (1980) and Hillman and Riley (1989), but extends it by allowing an additional winner-payment dimension, thereby bridging traditional auction theory and contest theory.

Understanding this hybrid auction format is important theoretically and practically. Theoretically, it generalizes the contest model pioneered by Siegel (2009, 2010) and others, providing a more granular view of bidder behavior when two types of expenditures are possible. Notably, complete-information all-pay auctions, such as those in Baye et al. (1996), emerge as a special case of our model. By studying a broader class of two-dimensional bidding games, we can capture phenomena that neither pure contests nor standard auctions alone can explain, such as discontinuities in bidding strategies and payoffs that arise when bidders switch between effort and monetary bids.

Practically, many real-world allocation problems exhibit exactly this hybrid structure. For example, in public procurement of infrastructure or services, bidders often compete on price and quality or technical merit, which is evaluated via a scoring rule. Such scoring auctions effectively require firms to incur up-front costs to improve quality, representing a sunk effort, while also offering a price bid paid if they win. Similarly, radio spectrum license auctions frequently impose coverage or investment obligations on winners in addition to monetary bids. In these cases, telecom firms not only pay for the license but also commit to costly network rollouts. Our hybrid auction framework closely mirrors this blend of winner-pay bids and all-pay commitments. Another example is online advertising auctions, where platforms like Google assign ad slots based on a combination of advertisers' bids and ad quality scores; the ad quality can be seen as an effort to create relevant ads, incurring cost regardless of winning, and the bid is paid by the winner per click. By modeling two-dimensional bids, our theory can thus shed light on allocation mechanisms in domains ranging from spectrum sales to online platforms and public project competitions.

This paper's contributions are threefold. First, we precisely situate our hybrid auction in the context of existing theory as a generalization of complete-information contests. Traditional auction theory often assumes private information and focuses on incentive-compatible mechanism design, as in Myerson (1981), or analyzes standard formats like first-price and second-price auctions. By contrast, our work aligns with the contest literature, specifically with models of asymmetric all-pay contests with two-stage investments. Recent studies by Siegel (2009, 2010) and by Melkonyan (2013) emphasized contests where each player's total bid has an unconditional part, paid irrespective of the outcome, and a conditional part, paid only if winning. Our model builds on this foundation, placing it within the broader auction theory landscape as a variant of a first-price all-pay auction. Indeed, if the conditional bid is set to *zero*, then our mechanism reduces to the standard all-pay auction studied by Hillman and Riley (1989) and by Baye et al. (1996), wherein all bidders' expenditures are sunk and the highest effort wins. Conversely, if the unconditional effort is *zero*, then our model becomes a classic first-price auction where only the winner pays. By encompassing both extremes, the hybrid auction resides in the intersection of these two subfields, extending each. It is essentially a multidimensional auction in the spirit of Che (1993) and Asker and Cantillon (2008), where bids have multiple components like price and quality.

Second, we highlight the new theoretical insights and methodological innovations our work offers relative to prior literature. One key distinction is that we drop certain continuity and convexity assumptions that earlier analyses relied on. Siegel's influential works on contests derived equilibrium payoffs under the assumption of continuous score functions and cost functions. In our more general hybrid auction, the score function that determines the winner can be nonconvex or piecewise linear, leading to discontinuous jumps in players' best responses. We show that the winning and losing payoff functions in Siegel's contest models can fail to be continuous in our setting; intuitively, a slight change in winning probability can cause a bidder to abruptly switch from prioritizing effort to money or vice-versa, creating a discontinuity in their payoff. As a result, existing results, like Siegel's closed-form equilibrium characterization, do not directly apply. We address this challenge by adopting a more fundamental, constructive approach to equilibrium analysis. Particularly, we explicitly derive players' optimal effort-bid combinations for various regions of the score function, rather than assuming a smooth equilibrium from the outset. This approach allows us to handle corner solutions and nonconvex strategy sets that were beyond the scope of previous models. We invoke results on equilibrium existence in discontinuous games to ensure that even with jumps in best responses, a Nash equilibrium exists in our hybrid auction. Methodologically, this represents a significant extension of contest modeling techniques.

We provide an algorithmic equilibrium construction analogous in spirit to Siegel's algorithm, but robust to discontinuities and tailored to compute players' equilibrium effort-bid strategies. Moreover, our model explicitly incorporates the possibility of an empty bid or inactive bidder, which is a scenario where a player, facing an extremely low win probability, chooses a minimal or *zero* bid without affecting their opponent's winning chance. We show that in equilibrium a player with virtually no chance of winning can still strategically choose an arbitrary losing bid, a nuance that does not arise in standard auctions. Accounting for these edge cases enriches the theory by describing strategic behavior in situations where one player is effectively out of contention, which is a concept not captured by earlier continuous models.

Third, we explore the applicability of our hybrid auction theory to practical allocation problems, underlining its relevance beyond the abstract model. Many real-world auction and contest mechanisms can be viewed as special cases of or inspirations for our setup. Communications regulators often auction spectrum licenses not purely to the highest monetary bidder, but using schemes that ensure network rollout or service quality. For example, regulators impose coverage obligations, such as requiring the winner to supply service to rural areas or meet roll-out targets, alongside financial bids. This effectively creates a hybrid: All bidders must be prepared to incur the cost of meeting these obligations, which is a sunk cost if they lose, while also competing via license payments. Our model's effort e can represent the costly fulfillment of coverage requirements, and the bid b represents the price paid for the license. The outcome being decided by a score $s(e, b)$ reflects how auctioneers often use a point system to balance price and nonprice attributes. By analyzing such a score auction as a game, our theory can inform the design of spectrum auctions to achieve policy goals like broad coverage without inducing excessively wasteful bidding. Indeed, theoretical analyses have noted that incorporating coverage commitments tends to reduce bidders' purely monetary overspending, aligning with our finding that hybrid formats can lower total expenditure.

Another notable example arises in the context of public procurement and contracting. Government and private procurers frequently use scoring auctions to evaluate bids on multiple criteria. Bidders propose, for instance, a price and a quality level or technical solution, and the highest combined score wins. A specific example is infrastructure procurement. Firms bidding to build a highway or bridge may be scored on price, technical quality, and completion time. Che's (1993) model of multidimensional auctions shows that if the buyer announces a scoring rule, then suppliers effectively face a tradeoff between offering a lower price and a higher quality. In our hybrid auction interpretation, the investment in quality or technology can be seen as an up-front cost—which the firm expends in preparing designs, prototypes, or meeting technical specs—akin to the effort e, whereas the price bid corresponds to the payment if the contract is won. Our framework thus maps onto these scenarios, suggesting how equilibrium bids might be formed. For instance, we predict that if quality has diminishing returns in score, then firms might not over-invest in quality but rather balance it with price, reflecting an equilibrium similar to what our model would yield. Understanding these tradeoffs helps ensure that procurement auctions select the bidder providing the best value for money, not just the cheapest or the highest quality in isolation.

Our analysis is framed within the extensive auction and contest literature. Early work on all-pay auctions assumed complete information and homogeneous bidders, yielding simple equilibrium predictions, as seen in Hillman and Riley (1989), Baye et al. (1993), and Che and Gale

(1998). In these models, the prize is awarded to the highest bidder, but all bidders incur their bids as costs. Siegel (2009, 2010) provides a definitive extension to asymmetric all-pay contests, deriving closed-form equilibrium payoffs even with heterogeneity. The survey by Kaplan and Zamir (2015) reviewed these and other developments in contest theory. In our setting, we depart from the usual all-pay format by allowing bidders all-pay and winner-pay investments, each affecting the winning probability.

Several papers have analyzed hybrid contest formats that mix all-pay and winner-pay investments. Haan and Schoonbeek (2003) studied a symmetric contest where each player expends a sunk effort and submits a bid payable only if they win. They showed a unique pure-strategy equilibrium in which all active players use the same bid and aggregate effort equals that bid. Melkonyan (2013) examined a symmetric n-player hybrid contest with constant elasticity of substitution or Cobb–Douglas production functions and a Tullock success function. In contrast, our model allows general weakly increasing production and contest success functions, relaxing these restrictions and yielding simpler existence conditions. Similarly, Siegel (2009, 2010) develops a broad framework for contests with both sunk and conditional investments. However, each player in his model effectively chooses a single investment level whose cost is exogenously split into sunk and contingent parts. By contrast, we endogenize effort and bid separately, so their cost shares can adjust with the economic environment. Moreover, Siegel's contest success function is perfectly discriminatory (highest effort always wins), whereas we allow stochastic winning probabilities via concave scoring rules.

Other related studies consider multiple effort dimensions or endogenous contest design. Gradstein and Konrad (1999) analyzed multistage (committee) contests, and Chen (2003) studied contests with sabotage efforts. Models of conflict allocate resources between production and appropriation (Hirshleifer 1991). Konrad and Kovenock (2005) allowed each contestant to split effort into multiple all-pay arms, with the overall winning probability aggregated. By contrast, our contest has a single performance dimension influenced by two investment channels. Likewise, the multi-attribute auction literature (e.g., Che (1993)) has examined bids on price and quality combined by a scoring rule, but typically in those models, costs are incurred only by the winner. On a similar note, typical winner-pay contests and contests with partial cost reimbursements (Matros and Armanios 2009) illustrate how relaxing the all-pay assumption alters equilibrium outcomes; however, we focus on the hybrid all-pay/winner-pay tradeoff.

Although our work is theoretical, hybrid auction models have practical applications in procurement, innovation tournaments, and political lobbying. Che and Gale (2003) investigated an R&D procurement contest where firms bid an innovation quality and a reward, which is effectively an all-pay investment plus a conditional prize claim. In lobbying, Baye et al. (1993) and Che and Gale (1998) showed that imposing bid caps or biases can raise total effort, and Szech (2015) noted that the finding that handicaps increase all-pay bidding holds under asymmetric tie-breaking. Our model, with its primitive effort and bid choices, provides a unifying framework that nests such applications while relaxing restrictive assumptions of prior work.

The remainder of the paper is organized as follows. Section 2 introduces the basic framework, detailing the players' actions, the scoring functions, and the payoff structure. Section 3 then grounds this abstract model in practice by examining several canonical score functions—namely, linear, entry-fee, and Leontief forms—and derives closed-form equilibria for each, thereby illustrating how the general theory adapts to particular functional forms. Section 4 turns to the core theoretical contribution: It develops two constructive algorithms that establish equilibrium existence and characterize strategic behavior even when best responses are discontinuous. Section 5 synthesizes these insights, presenting the principal results on equilibrium payoffs, multiplicity, and comparative statics, and highlighting the novel strategic phenomena that arise in hybrid auctions. Finally, Section 6 concludes with a discussion of the key findings and potential extensions. All proofs are relegated to the Appendix.

II. Model

We consider an auction with two players.

Players and Valuations: The players are risk-neutral and indexed by $i = 1, 2$. Each player i has a valuation $v_i > 0$ for the prize that can be thought of as the monetary value of winning.

Actions (Effort and Bid): Each player simultaneously and independently chooses two nonnegative quantities; an effort level $e_i \geq 0$ and a bid $b_i \geq 0$. The effort e_i is an unconditional (sunk) investment that the player must expend regardless of the outcome, whereas the bid b_i is a conditional payment that is paid only if the player wins the prize. In other words, e_i is an irreversible cost, representing an all-pay investment, and b_i is a winner-pay amount, akin to a standard auction bid. For example, in a procurement auction a firm might spend resources to prepare its proposal (an up-front cost e_i that is lost whether or not the firm wins) and submit a price quote b_i, which it will effectively pay only if it wins the contract. Each player's objective is to maximize their expected payoff.

Remark: This setting extends the contest framework of Siegel (2009, 2010). In Siegel's formulation, players effectively choose a score directly, and their costs differ depending on whether they win or lose (which implicitly allows two kinds of investment). Our approach instead has players explicitly choose two forms of investment, an all-pay effort e_i and a winner-pay bid b_i, which is a more primitive and transparent representation. This more fundamental setup does not require certain technical assumptions that Siegel's model needed (for instance, we do not assume cost functions are continuous in the score), and it accommodates a broader class of score functions.

Score Function and Outcome: Each player's effort and bid are combined into a single score through a score function $s_i(e_i, b_i)$. Formally, for each player i, we have a scoring function $s_i : \mathrm{R}^2_+ \rightarrow \mathrm{R}$, such that $s_i(e_i, b_i)$ is the score achieved by player i when they choose effort e_i and bid b_i. We assume that each $s_i(e_i, b_i)$ is weakly increasing in each argument, e_i and b_i (i.e., $s(e_i', b_i') \geq s(e_i, b_i)$ whenever $e_i' \geq e_i$ and $b_i' \geq b_i$). We also normalize $s_i(0, 0) = 0$ (i.e., *zero* effort and *zero* bid yields *zero* score). The winner of the auction is determined by these scores: The player with the higher score wins the prize. In case of a tie (both players obtain the same score), the winner is chosen randomly with equal probability. The tie-breaking rule does not affect the results, so we adopt this simple rule purely for convenience.

Payoffs: The payoff functions in this hybrid auction reflect the two types of investments. Suppose that player i chooses effort e_i and bid b_i. First, if he wins the prize, then he pays his bid b_i in addition to having already paid the effort e_i. Thus, the winner's net payoff is $v_i - e_i - b_i$, which is the value of the prize minus total expenditures. Second, if he loses, then he pays nothing further besides the sunk effort e_i. The loser's net payoff is $0 - e_i$ $(= -e_i)$, since he gets no prize and still incurs the effort cost. In general, if a player's probability of winning is p, then his expected payoff is: $pv_i - pb_i - e_i$, because the bid b_i is only paid in the winning scenario (with probability p), whereas the effort e_i is paid with certainty regardless of outcome.

Remark: We assume a linear cost structure only for simplicity; one unit of effort or bid costs one unit of payoff. However, any strictly increasing cost functions for effort and bid can be absorbed into the score function via a change of variables. Particularly, suppose more generally that a player had a cost of $c_e(e_i)$ for his effort e_i and a cost of $c_b(b_i)$ for his bid b_i, with $c_e(\cdot)$ and $c_b(\cdot)$ strictly increasing. We could define a modified score function $\tilde{s}_i$ in terms of effective effort and bid, $\tilde{s}_i\left(c_e(e_i), c_b(b_i)\right) = s_i(e_i, b_i)$, effectively measuring e_i and b_i in cost units. The player would then choose effort and bid in units of cost; and in this transformed model with the modified score function $\tilde{s}_i$, the cost becomes linear. Thus, any outcome in a general cost setting can be replicated in our linear-cost model by an appropriate redefinition of $\tilde{s}_i$.

Reach (Maximum Attainable Score): Each player i is constrained in how high a score they can achieve by their valuation v_i, since they would not rationally spend more than v_i in total (otherwise, their net payoff would be negative even if they win). Following Siegel (2010), we define player i's reach (r_i), as the maximum score they can attain when expending up to their valuation (i.e., $r_i = \sup\{s_i(e_i, b_i): v_i \geq e_i + b_i\}$. In other words, r_i is the maximum score player i can attain without spending more than v_i in combined effort and bid. If an argmax exists (e.g., $(\overline{e}_i, \overline{b}_i)$), then we have $s_i(\overline{e}_i, \overline{b}_i) = r_i$. We assume without loss of generality (simply as a labeling convention) that $r_1 \geq r_2$ (i.e., player 1 has a weakly higher reach than player 2).

Timeline and Nash Equilibrium: The interaction can be summarized as a one-shot game. First, players know their valuations and learn their opponent's. Next, they simultaneously choose their effort levels and bids (e_1, b_1) and (e_2, b_2). Finally, a winner is determined based on the resulting scores. All aspects of the game are common knowledge. We employ Nash equilibrium as our solution concept: In equilibrium, each player's strategy maximizes their expected payoff given the other player's strategy, and no one can profit by unilaterally deviating. Generally, the equilibrium may involve mixed strategies, although in some cases a pure-strategy equilibrium exists.

III. Various Score Functions

Having laid out the general hybrid auction model in the previous section, we now turn to several concrete score functions that map a bidder's effort and bid into a single ranking metric. Examining specific functional forms—ranging from nearly perfect substitutes to strict complements—allows us to illustrate how the model's key forces operate in practice and highlight the conditions under which equilibrium is unique or exhibits multiplicity. Throughout this section, we assume that both bidders face the same score function, such that $s_1(\cdot, \cdot) = s_2(\cdot, \cdot)$. This keeps the algebra transparent while still capturing the full spectrum of strategic behavior generated by different degrees of substitutability.

A. Linear Score Function

Suppose the score is a linear combination of effort and bid: $s(e_i, b_i) = \alpha e_i + \beta b_i$, with some reals $\alpha > \beta$. If instead $\alpha \leq \beta$, then effort is weakly less effective than bidding, and the problem reduces to a standard single-dimensional first-price auction with no effort component.

> Equilibrium Characterization: In a hybrid auction with the linear score function $s(e_i, b_i) = \alpha e_i + \beta b_i$, there exist uncountably many equilibria. In every equilibrium, Player 2's expected payoff is *zero*, whereas Player 1's expected payoff can range from *zero* to $v_1 - v_2$. Particularly, two types of equilibria exist. In one equilibrium, Players 1 and 2 play the game according to the cumulative probability functions $F_1 : \mathrm{R}_+^2 \rightarrow [0, 1]$ and $F_2 : \mathrm{R}_+^2 \rightarrow [0, 1]$, respectively, such that $F_1(e_1, 0(= b_1)) = 0$ for $e_1 \in \left[0, \frac{\beta v_2}{\alpha}\right)$ and $F_1(e_1, 0(= b_1)) = e_1/v_2$ for $e_1 \in \left[\frac{\beta v_2}{\alpha}, v_2\right]$; and $F_2\left(0\left(=e_2\right), v_2\left(=b_2\right)\right) = \left(v_1 - v_2 + \frac{\beta v_2}{\alpha}\right)/v_1$, $\lim_{b_2 \to v_2} F_2\left(0\left(=e_2\right), b_2\right) - F_2\left(0, v_2\right) = 0$, and $F_2(e_2, 0(= b_2)) = (v_1 - v_2 + e_2)/v_1 - \left(v_1 - v_2 + \frac{\beta v_2}{\alpha}\right)/v_1$ for $e_2 \in \left(\frac{\beta v_2}{\alpha}, v_2\right]$. Second, in the other equilibrium, Player 1 expends $e_1 = \mathrm{B}/\alpha$, such that $v_1 \geq \mathrm{B}/\alpha \geq v_2$ and bids $b_1 = 0$ with probability *one*, and Player 2 plays the game according to the cumulative probability function $F_2 : \mathrm{R}_+^2 \rightarrow [0, 1]$, such that for $b_2 \in [0, B/\beta]$, $F_2(0(= e_2), b_2) \leq (v_1 - B/\alpha)/(v_1 - b_2)$ if $b_2 < b_2^*$, $F_2(0(= e_2), b_2) \leq (v_1 - B/$

$\alpha + \beta b_2)/v_i$ if $b_2 \geq b^*_2$, $F_2(0(= e_2), B/\beta(= b_2)) = 1$, and $\lim_{b_2 \to B/\beta} F_2\left(0\left(=e_2\right), b_2\right) - F_2\left(0, B/\beta\right) = 0$, where $b^*_2 = \sup\{b_2 : F_2(0(= e_2), b_2) < \beta/\alpha\}$. The second type of equilibria generalize Algorithm 1, which will be presented in Section 4.

In the first kind of equilibria, both players mix strategies over their actions; Player 1 randomizes over a range of effort levels by bidding *zero* with probability *one*, and Player 2 randomizes over a range of bid amounts by exerting *zero* effort. In this case, Player 1's expected payoff is $v_1 - v_2$, and Player 2's is *zero*, which is an outcome analogous to a standard all-pay auction. In the second equilibrium, Player 1 uses a pure strategy by choosing a fixed positive effort level and bidding *zero*, whereas Player 2 employs a mixed strategy over bids up to a corresponding threshold. In these equilibria, Player 1 wins with certainty but incurs a higher cost. Consequently, Player 1's net payoff is lower than $v_1 - v_2$; indeed, any value between 0 and $v_1 - v_2$ can be achieved by an appropriate choice of Player 1's effort, whereas Player 2's remains *zero*.

Remark: The linear score case highlights how a lower-valuation bidder can attempt to compensate for a valuation disadvantage by relying on the bid component. For instance, imagine a contest where the final score is a weighted sum of a contestant's project quality (which requires a costly effort) and a monetary bid (paid only if the contestant wins). Even though effort has a higher weight in the score (i.e., $\alpha > \beta$), the weaker player (Player 2 in our context) may try to bid high (offer a large b_2) while putting in minimal effort ($e_2 = 0$). This strategy constitutes a threat that is costly only in the event of winning. In equilibrium, the higher-valuation player (Player 1 in our context) responds by randomizing her effort level to make Player 2 indifferent. The result, in the first type of equilibria above, is analogous to an all-pay auction: Player 1 wins the prize with probability one in expectation and secures an expected surplus of $v_1 - v_2$, whereas Player 2's expected payoff is *zero*.

Alternatively, Player 1 can deter Player 2 entirely by committing to a sufficiently large effort with no bid, ensuring a win with probability *one*. This is the second type of equilibrium. In that case, Player 1 effectively over-invests in effort, guaranteeing victory but dissipating part of her valuation advantage in cost. As a result, his realized payoff is lower than the maximum $v_1 - v_2$. This continuum of equilibria—from the all-pay outcome to the deterrence outcome—illustrates that although Player 1 always wins, he can choose how much rent to leave on the table for himself by adjusting his strategy. Importantly, the linear score example also demonstrates the role of discontinuities in players' best responses. Because Player 2's optimal strategy can involve a sudden switch from making only a conditional investment (bidding) to making an unconditional investment (effort) at certain thresholds, the resulting payoff functions are not continuous. Such discontinuities cannot be handled by the methods of Siegel (2009, 2010), but our hybrid auction framework accommodates them and finds the equilibria described above.

B. First-price Auction with an Entry Fee

Now consider a score function that effectively imposes a minimum effort requirement, which is an entry fee, to participate in bidding. Let $\underline{e}$ (≥ 0) be a required effort level or cost, and define:

$$\begin{aligned} s_i(e_i, b_i) &= b_i \text{ if } e_i \geq \underline{e} \text{ and} \\ &= 0 \text{ if } e_i < \underline{e} \end{aligned}.$$

This structure ensures that a player must expend at least $\underline{e}$ effort; otherwise, his score is treated as negligibly low, preventing him from winning. In other words, a bidder who does not pay the entry cost $\underline{e}$ is disqualified from winning regardless of their bid.

> Equilibrium Characterization: In a hybrid auction with an entry fee $\underline{e}$, uncountably many equilibria exist. In every equilibrium, Player 1's expected payoff is $v_1 - v_2$, and Player 2's expected payoff is *zero*. Moreover, Players 1 and 2 play according to the cumulative probability functions $F_1 : \mathrm{R}^2_+ \rightarrow [0, 1]$ and $F_2 : \mathrm{R}^2_+ \rightarrow [0, 1]$, respectively, such that $F_1(e_1, b_1) = 0$ for $e_1 < \underline{e}$ and $F_1(\underline{e}(= e_1), b_1) = \underline{e} / (v_2 - b_1)$ for $b_1 \in [0, v_2 - \underline{e}]$; and $F_2(0(= e_2), \infty) = (v_1 - v_2 - \underline{e}) / v_1$ and $F_2(\underline{e}(= e_2), b_2) = (v_1 - v_2 - \underline{e}) / (v_1 - b_2)$ for $b_2 \in [0, v_2 - \underline{e}]$.

Remark: The strategic logic here is similar to the first kind of equilibria in the linear score case. If a bidder must pay a cost to enter the auction, the lower-valuation player, namely, Player 2, will be deterred from aggressive competition. In equilibrium, Player 1, who holds the higher valuation, always pays the entry cost $\underline{e}$ and participates, whereas Player 2 may randomize between entering—paying $\underline{e}$ and bidding—and staying out by not paying the cost. Any equilibrium outcome entails Player 1 winning the prize and extracting the full value difference $v_1 - v_2$ as his expected payoff, whereas Player 2 ends up with *zero*. Intuitively, the entry fee forces Player 2 to incur a definite cost in order to compete; this requirement, given the valuation disadvantage, prevents Player 2 from obtaining any surplus. A real-world interpretation is a first-price auction where bidders must pay a non-refundable deposit or undergo a difficult qualification process to participate. The stronger bidder can always afford to pay this fee and outbid the weaker one, so the weaker bidder either drops out or participates only in a limited way that does not yield a positive payoff.

C. Leontief Score Function

Finally, consider an extreme case of perfect complements. Let $A > 0$, $\alpha > 0$, and $\beta > 0$, and define

$$s_i(e_i, b_i) = A \cdot \min\{\alpha e_i, \beta b_i\}.$$

In this Leontief score function, effort and bid are effective only in proportion to each other; a player cannot substitute one input for the other beyond the fixed ratio $\alpha : \beta$. Particularly, if a player puts in *zero* effort (i.e., $e_i = 0$), then additional bidding does not increase the score, and the same limitation applies to effort when the bid is fixed.

> Equilibrium Characterization: In a hybrid auction with the Leontief score function $s_i(e_i, b_i) = A \cdot \min\{\alpha e_i, \beta b_i\}$, there exists a unique equilibrium, where Player 1's and Player 2's expected payoff is $v_1 - v_2$ and *zero*, respectively. Moreover, given a score $s \in R_+$, Players 1 and 2 play according to the cumulative probability functions $F_1 : R_+ \to [0, 1]$ and $F_2 : R_+ \to [0, 1]$, respectively, such that
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> $$F_1(s) = \frac{s}{\beta A \cdot v_1 - s} \cdot \frac{\beta}{\alpha} \quad \text{and} \quad F_2(s) = \frac{\alpha A(v_1 - v_2) + s}{\beta A \cdot v_1 - s} \cdot \frac{\beta}{\alpha}.$$
>
> Then, their equilibrium efforts and bids are defined as

$$e_i\left(F_j(s), s\right) = \frac{s}{\alpha A} \text{ and } b_i\left(F_j(s), s\right) = \frac{s}{\beta A},$$

Where $\{i, j\} = \{1, 2\}$.

Remark: Under the Leontief score case, a player must invest in both dimensions to improve their score, because lacking either sufficient effort or bid will cap the attainable score. Strategically, this environment mirrors a standard all-pay contest; in fact, it corresponds to the simple contest model of Siegel (2010) as a special case. The low-valuation player, Player 2, cannot mount any serious challenge without spending substantial resources on both fronts, which is not worthwhile given his valuation. Thus, a single equilibrium outcome exists: Player 1, who holds the higher valuation, wins the prize with certainty and earns an expected payoff of $v_1 - v_2$, whereas Player 2 earns *zero*. This outcome is identical to the classic result in contests, where the higher-valuation player secures all the surplus.

IV. Algorithms

In this section, we construct two algorithms to demonstrate equilibrium existence and characterize equilibrium behavior in the auction. The first algorithm provides a general construction that applies to any two-player auction, thereby proving the existence of at least one Nash equilibrium. For auctions with more than two players, the algorithmic construction becomes intractable; however, one can still establish existence through a general equilibrium existence theorem, such as Simon and Zame (1990). Importantly, the equilibrium induced by this first algorithm reveals how players' strategies in a hybrid auction differ from those in a typical single-investment auction. The second algorithm applies under specific conditions and yields an equilibrium with markedly different payoff implications, particularly highlighting cases where results from standard auctions—such as the equilibrium payoff structure in Siegel (2009, 2010)—do not carry over to the hybrid framework. Taken together, these two algorithms delineate when a hybrid auction's equilibrium coincides with that of a standard auction and when it deviates.

We initially introduce two key concepts that will be used in constructing the algorithms. Define the unconditional empty score as

$s_2(0, v_2)$, which is the score achieved by Player 2 if he expends *zero* effort ($e_2 = 0$) and bids his entire valuation ($b_2 = v_2$). Likewise, define the conditional empty score as $\lim_{b_2 \to \infty} s_2(0, b_2)$, which can be interpreted as the limiting score of Player 2 if he puts in *zero* effort but were to bid an arbitrarily large amount. These two benchmarks represent extreme strategies for Player 2 that guarantee him a payoff of *zero*. For instance, if Player 1's actions induce a sufficiently high score, then Player 2 can always respond by either bidding his value with no effort (yielding the unconditional score) or by making an extremely large bid with no effort (yielding the conditional empty score). In either case, Player 2's expected payoff becomes *zero*, since bidding one's full value or more ensures no positive surplus when winning. Thus, Player 2 is essentially indifferent (*zero* profit) when resorting to these empty strategies, and he has no incentive to deviate from them if they are part of an equilibrium. These properties shape the equilibrium in a hybrid auction and provide a primary reason why the hybrid auction outcomes differ from those in typical auctions: The availability of an unconditional empty or conditional empty strategy for the disadvantaged player changes how the advantaged player must strategize.

For each player i, we formalize the player's optimal behavior in achieving a given score. Let $e_i(s, p)$ and $b_i(s, p)$ be defined as follows: For a target score s and a probability p ($\in [0, 1]$) of winning, a pair ($e_i(s, p)$, $b_i(s, p)$) is an optimal effort-bid pair that attains at least score s (such that $s_i(e_i, b_i) \geq s$) while maximizing player i's expected payoff $p \cdot (v_i - b_i) - e_i$, where v_i is player i's valuation. In other words, given that player i wins with probability p, $e_i(s, p)$ and $b_i(s, p)$ are the effort and bid levels that achieve the required score s at a minimal cost (the winning probability p times the value minus bid minus effort cost). When the score function $s_i(\cdot, \cdot)$ is continuous in its arguments, such an optimal pair exists for any s and p. If multiple optimal pairs yield the same payoff, then we choose a convention for definiteness; in this case, let $b_i(s, p)$ be the maximum optimal bid and $e_i(s, p)$ the minimum optimal effort that attain score s at the optimum. This selection ensures that $e_i(s, p)$ and $b_i(s, p)$ are well-defined functions of s and p. Intuitively, these functions describe how player i can achieve a score of at least s most efficiently given the probability p of winning.

Before presenting the algorithms, we establish two lemmas about these optimal-response functions.

Lemma 1. *Suppose that each score function* $s_i(e_i, b_i)$ *is continuous in its arguments. Given any score* s *and positive probability* p, *suppose that either* $e_i(s, p) > 0$ *or* $b_i(s, p) > 0$, *or both.*

1) *If* $s' > s$, *then we have that* $p \cdot b_i(s, p) + e_i(s, p) < p \cdot b_i(s', p) + e_i(s', p)$.

2) *In addition, we attain* $s_i(e_i(s, p), b_i(s, p)) = s$.

Lemma 1 means that given any target score s and positive winning probability p, if the optimal solution for player i to achieve score s involves a positive investment (i.e., $e_i(s, p) > 0$ or $b_i(s, p) > 0$, or both), then for any higher target $s' > s$, the total expenditure $p \cdot b_i(s, p) + e_i(s, p)$ is strictly smaller than $p \cdot b_i(s', p) + e_i(s', p)$. In other words, raising the required score while keeping the winning probability p fixed necessitates a strictly greater total cost. The second assertion means that at the optimum for score s, player i exactly meets the score requirement, $s_i(e_i(s, p), b_i(s, p)) = s$. Particularly, no slack exists; that is, the chosen (e_i, b_i) achieves a precise score s.

Lemma 2. *Given a real* $\pi \in \left[0, \max\left\{0, v_i - b_i(r_2, 1) - e_i(r_2, 1)\right\}\right]$, *define a function* $p_i^{\pi} : \left[s_2(0, v_2), r_2\right] \rightarrow \left[0, 1\right]$ *implicitly such that*

$$p_i^{\pi}(s) \cdot \left\{v_i - b_i\left(s, p_i^{\pi}(s)\right)\right\} - e_i\left(s, p_i^{\pi}(s)\right) = \pi.$$

In this definition, $p_i^{\pi}(s)$ *denotes the probability, such that given a score* $s \in \left[s_2(0, v_2), r_2\right]$, *the probability* $p_i^{\pi}(s)$ *induces Player i's expected payoff* π. 1) *This function* $p_i^{\pi}(\cdot)$ *is well-defined.* 2) *In addition, if* $s' > s$, *then we attain* $p_i^{\pi}(s') > p_i^{\pi}(s)$.

Lemma 2 implies that for each player i and any desired baseline payoff π between 0 and the maximum payoff he can secure $\max\left\{0, v_i - b_i(r_2, 1) - e_i(r_2, 1)\right\}$ (where r_2 is Player 2's reach), we can implicitly define a winning probability $p_i^{\pi}(s)$ as a function of the target score s, such that $p_i^{\pi}(s) \cdot \left\{v_i - b_i\left(s, p_i^{\pi}(s)\right)\right\} - e_i\left(s, p_i^{\pi}(s)\right) = \pi$. Moreover, this probability function $p_i^{\pi}(s)$ is strictly increasing in s.

We are now ready to construct the equilibria.

Algorithm 1 (General Equilibrium Construction): Assume each score function $s_i(e_i, b_i)$ is continuous and strictly increasing in both arguments. Let r_2 be Player 2's reach, and define $\pi = v_1 - b_1(r_2, 1) - e_1(r_2, 1)$. By the definition of r_1, we have $\pi \geq 0$. This π will turn out to be Player

1's guaranteed equilibrium payoff. Then, we construct a strategy profile as follows.

- Player 1's Strategy: He randomizes his effort and bid to achieve scores ranging from the unconditional empty score up to Player 2's reach. Specifically, for each score level s in the interval $[s_2(0, v_2), r_2]$, Player 1 assigns some probability to actions that yield a score of (at most) s. Formally, Player 1 chooses $\left(e_1\left(s, p_1^{\pi}(s)\right), b_1\left(s, p_1^{\pi}(s)\right)\right)$, such that

  $$e_1(s_2(0, v_2), p_1^{\pi}(s_2(0, v_2))) \leq e_1(s, p_1^{\pi}(s)) \leq e_1(r_2, p_1^{\pi}(r_2)) \text{ and}$$
  $$b_1(s_2(0, v_2), p_1^{\pi}(s_2(0, v_2))) \leq b_1(s, p_1^{\pi}(s)) \leq b_1(r_2, p_1^{\pi}(r_2)),$$

  where the probability $p_1^{\pi}(s)$ is defined as in Lemma 2 (i.e., $p_1^{\pi}(s) \cdot \left\{v_1 - b_1\left(s, p_1^{\pi}(s)\right)\right\} - e_1\left(s, p_1^{\pi}(s)\right) = \pi$).

- Player 2's Strategy: Player 2 mixes between two types of actions. First, with some probability mass concentrated on the lower end, Player 2 bids near his full valuation with *zero* effort. Concretely, we choose a sufficiently small $\varepsilon > 0$ and let him bid uniformly on the interval $[v_2 - \varepsilon, v_2]$ (with $e_2 = 0$) with total probability $p_2^0(s_2(0, v_2))$, where $p_2^0(s_2(0, v_2)) \cdot \{v_2 - b_2(s_2\left(0, v_2\right), p_2^0(s_2(0, v_2)))\} - e_2(s_2\left(0, v_2\right)$, and $p_2^0(s_2(0, v_2))) = 0$. This means that Player 2 sometimes plays an unconditional strategy of bidding his value (or just below it) without exerting effort. Second, for the remaining probability, Player 2 randomizes over effort-bid combinations to attain higher scores up to his reach. For each score s in $[s_2(0, v_2), r_2]$, he chooses actions $\left(e_2\left(s, p_2^0(s)\right), b_2\left(s, p_2^0(s)\right)\right)$, satisfying

  $$e_2\left(s_2\left(0, v_2\right), p_2^0\left(s_2\left(0, v_2\right)\right)\right) < e_2\left(s, p_2^0(s)\right) \leq e_2\left(r_2, p_2^0(r_2)\right) \text{ and}$$
  $$b_2\left(s_2\left(0, v_2\right), p_2^0\left(s_2\left(0, v_2\right)\right)\right) < b_2\left(s, p_2^0(s)\right) \leq b_2\left(r_2, p_2^0(r_2)\right),$$

  where the probability $p_2^0(s)$ is defined as in Lemma 2 (i.e., $p_2^0(s) \cdot \left\{v_2 - b_2\left(s, p_2^0(s)\right)\right\} - e_2\left(s, p_2^0(s)\right) = 0$).

In plain terms, Algorithm 1 constructs an equilibrium where Player 1 randomizes his actions over a continuum of scores up to r_2, and Player 2 randomizes over a similar range. The distributions $p_1^{\pi}(s)$ and $p_2^0(s)$ are calibrated, such that Player 2 is indifferent, earning a *zero* payoff across his support, and Player 1 secures a constant payoff π across his support. By Lemma 2, we can find strictly increasing probability functions $p_1^{\pi}(s)$ and $p_2^{0}(s)$ that satisfy the indifference conditions for every score s in the interval. Consequently, under these strategies, neither player can profit by deviating. Player 1 already obtains his guaranteed payoff π at every point of his support, and Player 2 is held to a *zero* payoff; any deviation would yield a negative payoff because his equilibrium payoff is *zero* and any higher score attempt comes at an excessive cost. Thus, this strategy profile constitutes a Nash equilibrium of the hybrid auction.[1]

Remark: Player 2's bidding strategy has support only on a narrow interval at the top of his value range: He bids between $v_2 - \varepsilon$ and v_2, essentially his full valuation, with some probability, and never bids much less than v_2 in equilibrium. Moreover, he never expends effort when making those highest bids, relying purely on the bid to compete at the lower end of the score range. This behavior is distinct from typical auction equilibria. In a standard first-price auction with complete information, for instance, the lower-value bidder might bid substantially below his value in equilibrium because the higher-value bidder only needs to outbid them slightly. Here, however, the presence of the unconditional investment (effort) changes the strategic calculus; that is, Player 2 must sometimes bid up to his valuation just to ensure that Player 1 does not secure an easy win with low effort. The equilibrium characterized by Algorithm 1 thus highlights how the availability of two investment channels forces the lower-valuation player to top out his bid distribution at his value. This represents a phenomenon not seen in single-dimensional auctions, such as those in Siegel (2009, 2010).

Notably, if the two players are symmetric (i.e., $v_1 = v_2$ and $s_1 = s_2$), the above construction yields a symmetric equilibrium. In that special case, one can verify that setting $p_1^{0}(s)$ and $p_2^{0}(s)$ for all $s \in \left[s_2(0, v_2), r_2\right]$ and having each player randomize in the manner described by Algorithm 1 results in both players earning a *zero* payoff, as each is held indifferent. Thus, the general algorithm is consistent with the possibility of symmetric equilibria when players are identical. In the generic case, however, the equilibrium will be asymmetric and yields

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[1] The equilibrium presented in Algorithm 1 was originally introduced by Baye et al. (1996) and subsequently extended by Siegel (2009, 2010). For further details, please refer to these studies.

the payoff advantages described above—for example, {layer 1 securing $\pi > 0$ and Player 2 earning 0).

Algorithm 2 (Special-case Equilibrium): Suppose the score functions are continuous and strictly increasing, and the conditional empty score exceeds player 2's reach (i.e., $\lim_{b_2 \to \infty} s_2(0, b_2) > r_2$). Moreover, assume that to achieve any score s at or above r_2, Player 1 must exert some positive effort; formally, $e_1(s, 1) > 0$ for all $s \geq r_2$. Under these conditions, we can find a bid level B, such that $r_1 \geq s_2(0, B) \geq r_2$.

- Player 1's Strategy: He plays a pure strategy; specifically, he chooses an effort-bid combination $\left(e_1\left(s_2(0, B), 1\right), b_1\left(s_2(0, B), 1\right)\right)$ with probability *one*.
- Player 2's Strategy: He bids only in the interval $[B - \varepsilon, B]$, uniformly for a sufficiently small positive ε. That is, all of his strategy mass is on bids in the range $[B - \varepsilon, B]$ while expending *zero* effort, $e_2 = 0$. In other words, Player 2 always bids at an extremely high level—around B—and never exerts effort.

In this strategy profile, Player 1 deterministically aims for the score $s_2(0, B)$, whereas Player 2 randomizes his bid around B without investing in effort. The construction ensures that $s_2(0, B) \geq r_2$, which means that Player 2 cannot normally reach this score by an optimal use of v_2 on his own. However, Player 2's strategy in this equilibrium sometimes involves extremely high bids—near B)—that if Player 1 were to aim lower than $s_2(0, B)$, he could risk being beaten by the extreme bid from player 2. In equilibrium, Player 1's choice of $\left(e_1\left(s_2(0, B), 1\right), b_1\left(s_2(0, B), 1\right)\right)$ yields him a payoff of $v_1 - b_1 - e_1$ because he certainly wins against $s_2(0, B)$ or ties at that score and wins with a 50% chance; However, if we choose B, such that $s_2(0, B) = r_2$ or slightly above, then Player 1 effectively wins with certainty due to a tie-break or minimal advantage. Player 2's expected payoff in this equilibrium is *zero*. Although he bids up to B—which might exceed his valuation v_2) and thus would incur a loss if he won at those high bids—the probability of winning against player 1's fixed strategy is effectively zero or negligible (i.e., if $s_2(0, B) = r_2$, then any bid up to B only ties the score at best; if $s_2(0, B) > r_2$, then Player 2 still cannot exceed Player 1's score by effort since he uses none, so he never wins). Therefore, Player 2 is indifferent to his extreme bidding because he almost never wins and never has to pay the bid. Player 1 has no incentive to deviate because any lower action would give Player 2 a chance to win with a high bid, and any higher action would just lower his payoff without increasing his winning probability, as he is already essentially guaranteed victory. Thus, no one can profit by deviating, meaning that this strategy profile constitutes an equilibrium.

Remark: The equilibrium in Algorithm 2 is qualitatively different from equilibria in standard auctions. Here, Player 2 is essentially threatening to bid an enormous amount—far above his true value v_2—with probability one, forcing Player 1 to respond by choosing a safe but costly action by expending effort to guarantee a high score. In classical auction models, , such as those in Siegel (2010) or a standard first-price auction, a bidder would never bid above his valuation in equilibrium, nor would the high-value bidder typically use a pure strategy if facing the possibility of extreme bids by the opponent. However, in this hybrid auction scenario, the combination of conditional and unconditional investments allows such an equilibrium to exist. This starkly illustrates that results and intuitions from typical auctions may not directly apply. The weaker player (Player 2) can employ an aggressive bidding strategy—one that would be irrational in a one-dimensional setting—to constrain the stronger player's behavior, leading to an outcome where Player 1's payoff is lower than would be predicted by classical models.

Algorithms 1 and 2 demonstrate that a hybrid auction can sustain equilibria similar to and dramatically different from those in standard auctions. In some cases, as seen in Algorithm 1, the equilibrium resembles the structure of all-pay or first-price auction equilibria but with key modifications, such as the use of the opponent's value as a bidding cap for Player 2. In other cases, such as Algorithm 2, the equilibrium involves strategies that have no analog in typical auctions, like bidding above one's valuation with *zero* effort, countered by a pure strategy from the opponent. These constructions will be instrumental in deriving the main results of our model.

V. Results

We finally turn to the main results of our analysis. We initially establish conditions for the existence of equilibrium and emphasize their necessity. Then, we examine the equilibrium payoffs, highlighting how they differ from classical auction outcomes. Finally, we provide a partial characterization of equilibrium behavior, underscoring distinctive strategic features, particularly the roles of the unconditional and conditional empty scores.

A. Existence of Equilibrium

Our first result confirms that under reasonable conditions on the score function, a Nash equilibrium exists in a two-player auction.

Theorem 1 (Existence of Equilibrium). *If the score function is continuous and strictly increasing, then there exists at least one Nash equilibrium.*

Theorem 1 is proved constructively by Algorithm 1 in Section 4, which explicitly builds an equilibrium under the stated conditions. The validity of Theorem 1 relies on the strict monotonicity and continuity of the score functions. Examples 1 and 2 illustrate that these conditions are tight; specifically, we demonstrate that relaxing either condition can lead to the non-existence of an equilibrium.

Example 1 (No Equilibrium with Discontinuous Score). *Let* $v_1 = v_2 = 1$ *and suppose the score function is defined as* $s_i\left(e_i, b_i\right) = e_i + b_i$ *if* $b_i > 0$ *and* $s_i\left(e_i, b_i\right) = 0$ *if* $b_i = 0$. *This score function* $s_i(\cdot\,,\cdot)$ *has a discontinuity at* $b_1 = 0$, *representing a jump from 0 to* $e_i + b_i$ *for any* $b_i > 0$). *In this scenario, no Nash equilibrium exists. Specifically, if a player chooses a bid of 0, then the opponent can execute a profitable deviation by choosing an arbitrarily small bid* $\varepsilon > 0$ *(with no effort). This triggers a discrete score jump from 0 to* ε, *allowing the opponent to win the prize at a negligible cost. This counterexample demonstrates that the continuity of the score function is indispensable for guaranteeing the existence of an equilibrium.*

Example 2 (Non-strictly Increasing Score). *Let* $v_1 = 2$ *and* $v_2 = 1$, *and consider a score function that is continuous but not strictly increasing. For instance, define* $s_i(e_i, b_i)$ *as follows:* 1) *for* $e_i \in [0, 1/5]$, $s_i(e_i, b_i)=$, e_i *(where the score increases with effort initially); for* $e_i \in [1/5, 3/5]$, $s_i(e_i, b_i) = 1/5$ *(a flat plateau where additional effort does not increase the score); for* $e_i \in [3/5, 7/10]$, $s_i(e_i, b_i) = e_i - 2/5$ *(the score increases again, matching the plateau, and reaches 3/10 at* $e_i = 7/10$*);*

for $e_i \in [7/10, 4/5]$*,* $s_i(e_i, b_i) = 3/10$ *(another flat segment); and for* $e_i \in [4/5, \infty]$*,* $s_i(e_i, b_i) = e_i - 1/2$ *(the score increases again after* $4/5$*, with* $s_i(4/5, 0) = 3/10$ *continuing smoothly). In this setup, the score function is weakly increasing but not strictly increasing due to the flat portions. One can show that no equilibrium exists. The key intuition for this failure is that in equilibrium, each player prefers to choose a score level that makes the opponent indifferent across the scores in their strategy's support. However, on a flat segment, raising effort strictly increases cost while leaving the score unchanged; consequently, neither player is willing to place positive probability on effort levels within the plateau. As a result, the equilibrium strategy would have to "jump" from below the plateau to above it. Such a jump creates a discontinuity in marginal incentives. To justify this jump, the winning probability must increase suddenly when moving from just below to just above the plateau, such that the higher cost is compensated by a sufficiently higher chance of winning. Because both players avoid the plateau, no mass of opponent play exists around that region to generate the required sudden increase in winning probabilities. Hence, the model cannot sustain the indifference conditions needed for an equilibrium, and an equilibrium fails to exist.*

B. Equilibrium Payoffs

Equilibrium in the hybrid auction is not necessarily unique; there may be multiple equilibria leading to different payoff distributions between the two players. However, one robust result is that Player 2's expected equilibrium payoff is always *zero* under the standard assumptions of strictly increasing score functions s_i. This mirrors a familiar outcome in many auction models—for example, in a first-price auction, the lower-value player typically earns *zero* payoff—and it holds in our hybrid auction as well, as demonstrated in Proposition 1.

Proposition 1 (Player 2's Equilibrium Payoff). *If the score functions are strictly increasing, then in any Nash equilibrium of a hybrid auction, Player 2's expected payoff is zero.*

Under strict monotonicity, suppose by contradiction that Player 2 earns a positive expected payoff in some equilibrium. Thus, a range of outcomes exists where Player 2 wins the prize while paying an effort e_2 and bid b_2 that total less than his value v_2, leaving him a positive surplus. Player 1, observing this, could profitably deviate by slightly increasing his bid b_1 or effort e_1 in those situations to capture the prize. Since Player 1 values the prize at least as much as player 2, he could overtake Player 2's score at a cost marginally above what Player 2 paid, thereby winning and still netting a positive payoff. This deviation would destroy any positive rent for Player 2. Thus, in equilibrium, Player 2 cannot net a positive payoff, and the only consistent outcome is that Player 2's expected payoff is driven to *zero*.

Proposition 1 highlights that the disadvantaged player (Player 2) does not profit in equilibrium. This result is reminiscent of the "no-rent" outcome for the low bidder in standard auctions or all-pay contests under complete information. This result is robust across all equilibria; even if multiple equilibria exist, Player 2's payoff remains *zero* in each.

The role of our assumption that score functions s_i are strictly increasing warrants attention. If we relax this condition while still assuming continuity, Player 2 could earn a positive payoff in some equilibria. For instance, if the score function has an upper bound or saturates at some level, then neither player can unilaterally guarantee victory by exceeding that score, and they might end up sharing the surplus. Consider a simple case where the score is literally bounded, say, $s_i(e_i, b_i) = 0$ for all e_i and b_i (an extreme case of a completely flat, bounded score function). In that trivial scenario, any effort or bid is wasted because it does not change the outcome; both players always tie with a score of 0, and the prize is split with equal probability. The unique equilibrium is for both players to invest nothing, and each wins with probability 1/2, yielding each an expected payoff of $v_i/2$. This is an equilibrium where Player 2 gets a positive payoff, indeed the same as Player 1. More generally, if score functions have an upper bound or a plateau at high values, the players may end up randomizing in a way that both obtain some rent. The key takeaway is that an unbounded, strictly increasing score function creates intense competition that drives the low-value player's payoff to *zero*, whereas bounds or flat regions can soften competition and allow positive payoffs for both. Our next example illustrates this.

Example 3 (Positive Payoffs for Both Players). *Let* $v_1 = v_2 = 1$ *and suppose the score function is bounded above or trivial, such as in the extreme case* $s_i(e_i, b_i) = 0$ *for all inputs). In this scenario, as argued, an equilibrium where neither player spends anything exists, as effort or* *bidding does not improve the outcome, and each wins half their value. Consequently, each player's expected payoff is 1/2. This equilibrium is unique in this setup. This simple case shows that when the score function fails to be unbounded or strictly increasing, both players can enjoy positive payoffs in equilibrium. In more realistic terms, if a maximum achievable score exists due to technological limits, then the high-value player may not fully exploit his advantage, and the low-value player may secure some rent.*

We now examine Player 1's equilibrium payoff in detail. Unlike Player 2's payoff, which is pinned at *zero*, Player 1's payoff can vary across different equilibria. The next proposition characterizes the range of possible payoffs for Player 1, which depends on the relationship between Player 2's reach r_2 and the conditional empty score. Let $\bar{s}_2 = \max\left\{\lim_{b_2\to\infty} s_2\left(0, b_2\right), r_2\right\}$, which is the greater of *i*) Player 2's conditional empty score—the highest score he could approach by bidding arbitrarily high with *zero* effort—and *ii*) Player 2's reach r_2. In many cases, $\bar{s}_2$ will equal r_2 if investing his full value optimally yields a higher score than any pure bid can achieve, but if pure bidding can yield a higher score, then $\bar{s}_2$ exceeds r_2.

Proposition 2 (Range of Player 1's Equilibrium Payoffs). *Suppose the score functions are continuous and strictly increasing. Then, in any equilibrium, Player* 1*'s expected payoff* π_1 *satisfies*

$$\pi_1 \geq \max\left\{0, v_1 - e_1\left(\bar{s}_2, 1\right) - b_1\left(\bar{s}_2, 1\right)\right\}.$$

In other words, Player 1 *can guarantee at least* $\max\left\{0, v_1 - e_1\left(\bar{s}_2, 1\right) - b_1\left(\bar{s}_2, 1\right)\right\}$ *in any equilibrium. Furthermore, if* $\bar{s}_2 > r_2$*, then there exists an equilibrium in which Player* 1*'s expected payoff is as low as*

$$v_1 - e_1\left(r_2, 1\right) - b_1\left(r_2, 1\right).$$

Specifically, Player 1*'s payoff can be pushed down to (at most) the value difference after incurring the cost to just beat* r_2*. More precisely, when* $\bar{s}_2 > r_2$*, Player* 1*'s equilibrium payoff can take any value in the range*

$$\left[\max\left\{0, v_1 - e_1\left(\bar{s}_2, 1\right) - b_1\left(\bar{s}_2, 1\right)\right\}, v_1 - e_1\left(r_2, 1\right) - b_1\left(r_2, 1\right)\right].$$

If instead $\bar{s}_2 > r_2$*, then all equilibria yield the same payoff for Player* 1, that is $\pi_1 = v_1 - e_1\left(r_2, 1\right) - b_1\left(r_2, 1\right)$.

This result delineates a floor and, in some cases, a range for Player 1's payoff. The term $v_1 - e_1\left(\bar{s}_2, 1\right) - b_1\left(\bar{s}_2, 1\right)$ represents Player 1's net payoff when he chooses an action to attain score $\bar{s}_2$ with certainty, given that $e_1\left(\bar{s}_2, 1\right)$ and $b_1\left(\bar{s}_2, 1\right)$ are the effort and bid required for Player 1 to reach the score $\bar{s}_2$ with probability 1. If $\bar{s}_2$ is greater than r_2, then $\bar{s}_2$ corresponds to a score that Player 2 cannot match even with his entire value. In that case, one equilibrium is for Player 1 to not greatly exceed r_2, but rather to randomize in a way that sometimes forces Player 2 to consider using the conditional empty bid. The possibility of those extreme bids by Player 2 can shrink Player 1's payoff down to the lower bound. Indeed, Algorithm 2 constructed in the previous section is an example where $\bar{s}_2 > r_2$ and Player 1's payoff is driven to the lower end of that range (in that case, Player 1 ends up essentially getting v_1 − (cost to bear r_2), which is the upper bound of Player 2's capability, thereby minimizing Player 1's rent).

Conversely, the lower bound $\max\left\{0, v_1 - e_1\left(\bar{s}_2, 1\right) - b_1\left(\bar{s}_2, 1\right)\right\}$ indicates the minimum payoff Player 1 would ever get. Typically this will be a positive number (unless v_1 is extremely low relative to the cost of achieving $\bar{s}_2$ that the expression becomes negative, in which case 0 is the trivial floor since payoffs cannot be negative if a player could always choose to not participate and get 0). This lower bound arises because Player 1 can always effectively aim for the score $\bar{s}_2$. Any action of Player 2 that tries to push beyond $\bar{s}_2$ yields no additional benefit, since $\bar{s}_2$ is either the highest score Player 2 can threaten with a bid or his maximum reach. Thus, any strategy of Player 2 that induces scores above $\bar{s}_2$ is strictly dominated by a less costly strategy for Player 2. As a result, Player 1 need not ever spend more than the cost to reach $\bar{s}_2$ in equilibrium; because Player 2 cannot credibly force him above that threshold, doing so would be over-investing with no return. Therefore, across all equilibria, Player 1 is guaranteed at least the payoff corresponding to winning at $\bar{s}_2$ (or 0, whichever is higher if $\bar{s}_2$ is extremely costly to reach), thereby establishing the floor for π_1.

If $\bar{s}_2 > r_2$, then the conditional empty score equals Player 2's reach;

effectively, Player 2 has no additional threat from infinite bidding beyond what he can do with his value. In such a case, the equilibrium is essentially unique in payoff terms: Player 4 will end up with v_1 − (cost to bear r_2) in any equilibrium, which is analogous to the typical auction outcome where the higher-value player gets the surplus equal to the difference in values (if e_1 and b_1 at r_2 correspond to an action that only matches Player 2's maximum effort). If $\overline{s}_2 = r_2$, then Player 1 wins by barely overcoming Player 2's best effort, earning the difference as payoff.

However, if $\overline{s}_2 > r_2$, then multiple equilibria can arise. In some equilibria, Player 1 might do well, which occurs when Player 2 does not force him to spend as much; whereas in others, Player 2's potential to bid extremely high—the $\overline{s}_2$ threat—forces Player 1 to expend more resources, lowering his payoff toward the lower bound. Proposition 2 states that as long as $\overline{s}_2 = r_2$, an equilibrium where Player 1's payoff is at the low end of the range, as demonstrated by the construction in Algorithm 2, and in any equilibrium it cannot be lower than that bound. Thus, the range is fully characterized.

The takeaway from Proposition 2 is that the introduction of the second investment dimension (bids) can widen the spectrum of possible outcomes. In a typical single-dimensional auction, the outcome might be essentially fixed (e.g., Player 1 always gets $v_1 - v_2$ and Player 2 gets 0 in the unique equilibrium). In the hybrid auction, however, because Player 2 can sometimes use the "nuclear option" of extremely high bids—even though he cannot afford them in a standard sense, they are equilibrium threats that occur with *zero* probability of winning—Player 1's payoff need not equal $v_1 - v_2$; it could be lower, even approaching 0 if v_1 has to be spent in costly effort. Thus, the hybrid model allows for multiple equilibria with different payoff splits, depending on how aggressive Player 2's bidding strategy is.

C. Partial Characterization of Equilibrium Behavior

We conclude with a partial characterization of equilibrium strategy properties that sharply contrast hybrid auctions with classical auctions. The following propositions assume the score function is strictly increasing, ensuring we are in the well-behaved regime where an equilibrium exists.

First, we find that Player 1 will never waste effort or bid on producing a score below the unconditional empty score of Player 2.

Proposition 3. *In any equilibrium, Player 1 never plays an action that yields a score lower than* $s_2(0, v_2)$*, which is Player 2's unconditional empty score.*

Proposition 3 highlights a critical difference from standard auctions: Player 1, the higher-value player, maintains a minimum effort-bid threshold in equilibrium, namely, the opponent's empty bid score. In a typical first-price auction, the high-value bidder might optimally bid just slightly above the low-value bidder's bid, which could be arbitrarily low if the low-value bidder is indifferent. Here, however, because Player 2 can always secure a score of $s_2(0, v_2)$ by bidding v_2, Player 1 must ensure he at least meets that threshold. If he did not, then he would essentially leave money on the table for Player 2 to grab at *zero* cost. Thus, the hybrid auction forces the high-value player to never under-bid or under-invest below a certain floor, which is a feature not present in auctions without the two-tier investment.

We now turn to the existence of pure-strategy equilibria. In standard complete-information all-pay auctions, equilibrium typically involves mixed strategies. By contrast, the hybrid auction may admit pure strategies. The following proposition states a sufficient condition for a pure-strategy equilibrium. Recall that $\bar{s}_2 = \max\left\{\lim_{b_2 \to \infty} s_2\left(0, b_2\right), r_2\right\}$.

Proposition 4. *If* $\bar{s}_2 > r_2$ *and score functions are continuous and strictly increasing, then there exists an equilibrium in which Player 1 plays a pure strategy. That is, Player 1 deterministically chooses a particular effort* e_1 *and bid* b_1*.*

We contrast Proposition 4 with Siegel (2010) or other all-pay contest models. Siegel's equilibria typically involve continuous mixed strategies by both players. Our hybrid model, by introducing the possibility of an empty bid strategy, can produce an equilibrium where the high-value player plays pure—essentially taking a deterministic stance—which is a result not observed in Siegel's contest under similar conditions. Thus, the condition $\bar{s}_2 \geq r_2$, meaning the ability to win via pure bid without effort is at least as high as the opponent's maximal score, ensures that the high-value player can safely commit to a particular strategy.

Finally, we consider Player 2's behavior in equilibrium. One might wonder if Player 2 will always limit himself to scores $s \leq r_2$, as r_2 is his maximum achievable score given his valuation. Surprisingly, the answer is no: In some equilibria, Player 2 will choose actions that exceed r_2 in terms of score. Of course, he can only do this by relying on the conditional empty score, which involves bidding beyond what is normally rational.

Proposition 5. *If* $\overline{s}_2 > r_2$ *and score functions are continuous and strictly increasing, then there exists an equilibrium in which Player 2 occasionally takes actions that induce a score higher than his reach* r_2.

Recall that in Algorithm 2 in Section 4, Player 2's strategy was to bid within $[B-\varepsilon, B]$ (with $s_2\,(0, B) > r_2$) with probability 1, thereby effectively playing scores above r_2. In equilibrium, these extreme bids do not actually yield a victory for Player 2; they primarily serve to constrain Player 1's strategy, but they are used with non *zero* probability. This represents a striking difference from classical auctions. In a standard first-price auction, a bidder would never bid above his value v_i in equilibrium because winning with such a bid yields a negative payoff. Here, Player 2 does bid up to B, which could be greater than v_2, constituting an action that would harm him if it led to winning. However, because Player 1 anticipates these bids and adjusts his strategy, Player 2 ends up almost never winning when he uses those high bids. Thus, he incurs no actual loss, and the threat of those bids is what matters. Nonetheless, this presence is crucial. It exemplifies how the hybrid auction allows for equilibrium behavior that might seem irrational, such as overbidding, but is perfectly rational as it serves to make the opponent indifferent.

VI. Conclusion

This paper introduced and analyzed a general model of hybrid auctions in which two bidders simultaneously choose a two-dimensional strategy consisting of a non-refundable effort level and a conditional monetary bid. A flexible score function—which can capture varying degrees of substitutability or complementarity between effort and bid—determines the winner. Our main theoretical contributions establish conditions for equilibrium existence in this game. Specifically, by allowing effort and bid to be chosen independently, our model generalizes the frameworks of Siegel (2009, 2010) to a broader class of scoring rules and strategic settings.

We proved that, under standard regularity assumptions on the score function, the game admits an equilibrium; in many cases, equilibrium payoffs are unique. In symmetric settings with identical bidders, we derived explicit equilibrium strategies in which each bidder exerts the same effort and submits the same bid. These strategies depend on the relative weight the score function places on effort versus bidding. When bidders are asymmetric, the equilibrium can feature different efforts and bids. We characterized these asymmetric outcomes and demonstrated how heterogeneity affects equilibrium allocations and payoffs.

A key insight from our analysis is the role of substitutability and complementarity between effort and bids in shaping equilibrium outcomes. When effort and bid are complementary in the score function—meaning that high effort increases the marginal value of bidding—bidders tend to invest more heavily in both dimensions. The equilibrium in this case involves higher combined effort and bid levels, which generally increases the total surplus. By contrast, when effort and bid are substitutes in the score function, a bidder can partially substitute effort for bidding or vice versa, which tends to temper overall expenditure. This substitutability typically results in lower total contributions and can therefore reduce the total surplus compared to the complementary case.

The generality of our model yields several insights for mechanism design and policy. In procurement auctions and government tenders, decision-makers often combine price bids with qualitative effort or quality metrics when awarding contracts. Our results indicate that the scoring rule’s relative weighting crucially influences firms’ incentives. For example, a scoring rule that makes effort and bid complements encourages greater overall investment in quality and can enhance social welfare, whereas a rule emphasizing substitutability may limit excessive spending but potentially at the cost of lower quality levels. Similarly, in innovation or R&D contests where participants expend effort and also state required rewards, our framework suggests that contest designers can promote higher innovation outcomes by structuring rewards so that effort and bidding complement each other. More generally, any contest using a composite score to allocate prizes can draw on our findings to understand how strategic dimensions interact.

Overall, this study deepens the theoretical foundation of multidimensional auction and contest design. By establishing equilibrium existence and uniqueness and by characterizing how effort-bid substitutability affects outcomes, we provided guidance on how scoring rules and contest formats may be structured to achieve desired objectives. Future work could extend this framework to more than two bidders, to incomplete-information environments, or to dynamic contest settings. Nonetheless, the current model already highlights the importance of accounting for multiple strategic dimensions in auctions and contests, offering a richer perspective for economists and policy-makers designing effective mechanisms.

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Appendix: Proofs

Proof of Lemma 1. The first assertion uses the continuity of the score functions s_i. We consider two possible cases. First, consider the case in which we have $e_i(s', p) > 0$. Then, given that each s_i is continuous, there exists a nonnegative real e', such that $e' < e_i(s', p)$ and $s_i(e', b_i(s', p)) \geq s$. Hence, according to the definitions of the functions e_i and b_i, we have

$$p \cdot b_i(s', p) + e_i(s', p) > p \cdot b_i(s', p) + e' \geq p \cdot b_i(s, p) + e_i(s, p).$$

Second, consider the case in which we have $b_i(s', p) > 0$. We can then show that

$$p \cdot b_i(s', p) + e_i(s', p) > p \cdot b_i(s, p) + e_i(s, p),$$

which completes the proof of the first assertion.

The second assertion follows from the first assertion. Obviously, by the definition of the functions e_i and b_i, we have $s_i(e_i(s, p), b_i(s, p)) \geq s$. Hence, $s_i(e_i(s, p), b_i(s, p)) > s$ is impossible. Suppose by way of contradiction that $s_i(e_i(s, p), b_i(s, p)) > s$. Let $s_i(e_i(s, p), b_i(s, p)) = s'$. Then, according to the first assertion, we have $p \cdot b_i(s, p) + e_i(s, p) < p \cdot b_i(s', p) + e_i(s', p)$, which contradicts the definition of $e_i(s', p)$ and $b_i(s', p)$, because $e_i(s', p)$ and $b_i(s', p)$ were supposed to be the effort and bid levels that achieve the required score s at a minimal cost. This contradiction completes the proof.

Proof of Lemma 2. The first assertion that the function $p_i^{\pi}(\cdot)$ is well-defined follows from the maximum theorem. Define a function $M : \mathrm{R}_+ \times [0, 1] \to \mathrm{R}$ as follows:

$$M(s, p) = \max_{(e_i, b_i) \in \mathrm{R}_+^2} p \cdot (v_i - b_i) - e_i \text{ such that } s_i(e_i, b_i) \geq s.$$

In this definition of the function M, we can replace the domain of (e_i, b_i) with a sufficiently large compact set while preserving the same results. Hence, according to the maximum theorem, $M(s, p)$ is continuous in s and p.

The following fact leads to the result that the function $p_i^{\pi}(\cdot)$ exists.

First, for any $p_i^\pi(\cdot)$, we have

$$M(s,0) = 0 \cdot \{v_i - b_i(s,0)\} - e_i(s,0) \leq 0 \leq \pi,$$

and

$$M(s,1) = 1 \cdot \{v_i - b_i(s,1)\} - e_i(s,1) \geq v_i - b_i(r_2,1) - e_i(r_2,1) = \pi,$$

where the last inequality comes from the facts that $s \leq r_2$; thus, $-b_i(s,1) - e_i(s,1) \geq -b_i(r_2,1) - e_i(r_2,1)$. Therefore, given that the function $M(s, p)$ is continuous in p, there exists a real p^*, such that $p^* \in [0, 1]$ and $M(s, p^*) = \pi$; hence, p^* can be $p_1^\pi(s)$, that is, $p_1^\pi(s) = p^*$.

To prove the second assertion, note that given $v_2 > 0$, we have $b_i(r_2,1) + e_i(r_2,1) = v_2 > 0$. Thus, we obtain $\pi \leq \max\{0, v_i - b_i(r_2, 1) - e_i(r_2, 1)\} < v_i$. Given $\pi < v_i$, we must have either $e_i(s, p) > 0$ or $b_i(s, p) > 0$, or both. Then, according to Lemma 1, we have

$$\begin{aligned}
& p_i^\pi(s) \cdot b_i(s', p_i^\pi(s)) + e_i(s', p_i^\pi(s)) > p_i^\pi(s) \cdot b_i(s, p_i^\pi(s)) + e_i(s, p_i^\pi(s)) \\
\Leftrightarrow\ & p_i^\pi(s) \cdot \{v_i - b_i(s', p_i^\pi(s))\} - e_i(s', p_i^\pi(s)) < p_i^\pi(s) \cdot \{v_i - b_i(s, p_i^\pi(s))\} - e_i(s, p_i^\pi(s)) \\
\Leftrightarrow\ & M(s', p_i^\pi(s)) = p_i^\pi(s) \cdot \{v_i - b_i(s', p_i^\pi(s))\} - e_i(s', p_i^\pi(s)) < M(s, p_i^\pi(s)) = \pi \\
\Leftrightarrow\ & M(s', p_i^\pi(s)) < \pi.
\end{aligned}$$

Recall that from the definition of the function $M(\cdot,\cdot)$, we can see that the function $M(s, p)$ is strictly increasing in p. Therefore, from the equation $M(s', p_i^\pi(s')) = \pi$, we obtain $p_i^\pi(s') > p_i^\pi(s)$, which then completes the proof.

Proof of Theorem 1. The result follows directly from the construction provided in Algorithm 1 in Section 4.

Proof of Proposition 1. The proof is omitted as it follows the same logical lines as the proof of Theorem 1 in Siegel (2009).

Proof of Proposition 2. The result follows directly from Algorithm 2 in Section 4, which explicitly constructs an equilibrium yielding payoffs for Player 1 that coincide exactly with the range specified in Proposition 2.

Proof of Proposition 3. Suppose for the sake of contradiction that in some equilibrium, Player 1 chooses an action with positive probability resulting in a score s that is less than $s_2(0, v_2)$. Then, Player 2 could profitably deviate: Instead of playing his equilibrium strategy, he could simply bid slightly below v_2 without exerting effort, thereby achieving a score marginally below $s_2(0, v_2)$, which is still above s because $s < s_2(0, v_2)$ by assumption. By doing so, Player 2 would win the prize whenever Player 1 chose that low score s. Because his bid is below v_2, he would earn a positive payoff in those cases. Since this would happen with positive probability (whenever Player 1 uses $s < s_2(0, v_2)$), Player 2 would gain a positive expected payoff from this deviation. However, Proposition 1 established that in equilibrium, Player 2's payoff must be *zero*. Therefore, such a profitable deviation for Player 2 cannot exist; hence, Player 1's equilibrium support cannot include any score below $s_2(0, v_2)$.

Proof of Proposition 4. If $\bar{s}_2 > r_2$, then Player 1 can choose to play the deterministic strategy that yields score $\bar{s}_2$ with certainty. Given that $\bar{s}_2$ is at least r_2, Player 2 cannot outscore $\bar{s}_2$ without exceeding his own capacity or resorting to an infinite bid, which would never be used with positive probability due to the negative payoff it entails upon winning. One can then construct an equilibrium where Player 2 randomizes in a way that makes Player 1 indifferent to slightly higher or lower actions around that pure strategy; effectively, Player 1 need not randomize and can stick to the score $\bar{s}_2$. Algorithm 2 provides an example (for the case $\bar{s}_2 > r_2$) where Player 1's strategy is pure.

Proof of Proposition 5. The result follows directly from the constructive equilibrium presented in Algorithm 2 in Section 4.

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