Identifying Fixed-effects Models with Heterogeneous Coefficients

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This review provides an introduction to identification and estimation methods for fixed-effects models with heterogeneous coefficients, which require identification strategies that are notably different from those for standard fixed-effects models. The strategies imply consistent estimation methods for the parameters of interest, which are also different from those used in standard fixed-effects models. As an introductory review, this work defers detailed implementation procedures for the estimation methods to future studies.

Keywords: Fixed effects, Difference-in-differences models, Panel data models, Unobserved heterogeneity

JEL Classification: C23

I. Introduction

Fixed-effects models are widely used in many empirical applications involving panel data. Such models are useful, because they allow researchers to account for unobserved heterogeneity across units or time periods. Fixed-effects models are typically implemented with homogeneous coefficients, which implies that the effects of the regressors on the outcome variable are the same across all the units.

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[**Seoul Journal of Economics** 2025, Vol. 38, No. 1]
DOI: 10.22904/sje.20254.38.1.004

This implication is in contrast to the fact that the fixed effects in such models allow for unit-specific intercepts.

Recently, the literature on fixed-effects models that allow for not only heterogeneous intercepts but also heterogeneous coefficients in the context of o ne-way and two-way fixed-effects (TWFE) models has grown. This research includes studies on random coefficient models and difference-in-differences (DID) models that can account for treatment effect heterogeneity.

In this review, I provide a concise introduction to the methods. Given that the identification and estimation techniques for the models differ significantly from those used in standard fixed-effects models, I focus on the key ideas for identifying and estimating the parameters of interest. I defer the discussion of detailed implementation procedures for the estimation of the parameters to future research.

I begin by discussing the identification of one-way fixed-effects models with heterogeneous coefficients, which are commonly referred to as random coefficient models. The identification strategy for such models varies depending on whether the regressor is strictly exogenous or sequentially exogenous, in which dynamic regressors, such as lagged outcome variables, are permitted in the latter case. For the strictly exogenous case, I review the identification methods proposed by Chamberlain (1992) and Arellano and Bonhomme (2012), which involve time-series ordinary least squares (OLS) estimation. For the sequentially exogenous case, I discuss the partial identification method proposed by Lee (2022).

Then, I turn to TWFE models with heterogeneous coefficients, which have been extensively examined in the context of DID models with treatment effect heterogeneity. From the substantial body of literature, I focus on the methods proposed by Wooldridge (2021) and Borusyak, Jaravel, and Spiess (2024). The former used TWFE regression with interacted regressors, whereas the latter conducted imputation of the baseline values.

The remainder of this paper is organized as follows: Section II discusses one-way fixed-effects models, Section III expounds on TWFE models and emphasizes their application in DID contexts, and Section IV concludes the review.

II. Fixed-effects models with heterogeneous coefficients

In this section, I review one-way fixed-effects models with heterogeneous coefficients. For comparison, a standard one-way fixed-effects model is written as

$$Y_{it} = \alpha_i + X'_{it}\beta + \varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T, \tag{1}$$

where Y_{it} is the outcome variable, X_{it} is the vector of the regressors, α_i is the fixed effects, β is the vector of the coefficients, and ε_{it} is the idiosyncratic error term. I discuss this model in the context of short panel data, which corresponds to the asymptotics, where N tends to infinity, but T is fixed.

Depending on the context of the empirical application, I assume that X_{it} is strictly exogenous, sequentially exogenous, or a combination of both. X_{it} is said to be strictly exogenous if

$$\mathbb{E}(\varepsilon_{it} \mid X_{i1}, \dots, X_{iT}) = 0 .$$

Under this assumption, β can be estimated consistently by using the well-known within transformation estimator. Strict exogeneity implies that the error term ε_{it} is uncorrelated with the entire history of X_{it}. Consequently, the current error term ε_{it} is not allowed to be correlated with future values of X_{it}, which can effectively rule out the feedback effects of current outcome Y_{it} on future values of X_{it}. For example, strict exogeneity will be violated when X_{it} includes the lagged outcome Y_{it-1}.

In contexts where strict exogeneity is not plausible, such as when X_{it} includes a lagged outcome variable, sequential exogeneity can be considered. X_{it} is said to be sequentially exogenous if

$$\mathbb{E}(\varepsilon_{it} \mid X_{i1}, \dots, X_{it}) = 0 .$$

The assumption implies that ε_{it} is uncorrelated only with the current history of X_{it}, which allows the current error term ε_{it} to be correlated with future values of X_{it}. Under sequential exogeneity, the within transformation estimator is no longer consistent for β. Instead, β can be estimated consistently by using instrumental variable regression techniques, as pointed out by Anderson and Hsiao (1982), Arellano and Bond (1991), and Blundell and Bond (1998).

Next, I introduce a one-way fixed-effects model with heterogeneous coefficients, in which I replace β in (1) with β_i, as follows:

$$Y_{it} = \alpha_i + X'_{it}\beta_i + \varepsilon_{it}.$$

This model is commonly referred to as a random coefficient model in the literature. The specification allows intercept α_i and coefficient vector β_i to be unit specific. For notational simplicity, I define $\gamma_i \equiv (\alpha_i, \beta_i)'$ and $W_{it} \equiv (1, X'_{it})'$ and rewrite the above model as

$$Y_{it} = W'_{it}\gamma_i + \varepsilon_{it}. \tag{2}$$

The analysis of random coefficient models requires modifications to the setup of standard fixed-effects models. First, because γ_i is unit specific, a parameter of interest that appropriately summarizes the different values of γ_i across the units must be defined. In this review, I focus on the average value of γ_i and define the parameter of interest as

$$\theta = \mathbb{E}(\gamma_i).$$

Following the random coefficient model framework, I consider γ_i as a random variable and do not make any assumptions on its distribution. Then, I interpret the fixed effects, that is, the individual values of γ_i for each i, as random draws from the nonparametric distribution of γ_i. Similar to standard fixed-effects models, the random variable γ_i can be freely correlated with W_{it}. Then, I consider the expectation of γ_i as the parameter of interest.

Second, the identification of random coefficient models requires strong conditions for the strict and sequential exogeneity of X_{it}. In random coefficient models, X_{it} is said to be strictly exogenous if

$$\mathbb{E}(\varepsilon_{it} \mid \gamma_i, W_{i1}, \ldots, W_{iT}) = 0. \tag{3}$$

Similarly, X_{it} is said to be sequentially exogenous if

$$\mathbb{E}(\varepsilon_{it} \mid \gamma_i, W_{i1}, \ldots, W_{it}) = 0. \tag{4}$$

In other words, strict and sequential exogeneity in random coefficient models will require the error term ε_{it} to also be uncorrelated with γ_i. This additional conditioning is necessary, because γ_i is a random variable in the models.

Subsequently, I discuss the identification and estimation of $\mathbb{E}(\gamma_i)$ in the random coefficient models. I begin with a case where X_{it} is strictly exogenous, that is, when (3) holds. The model has been analyzed by Chamberlain (1992) and Arellano and Bonhomme (2012). For each i, I consider model (2) across its time series, as follows:

$$\begin{aligned} Y_{i1} &= W'_{it}\gamma_i + \varepsilon_{i1}, \\ Y_{i2} &= W'_{it}\gamma_i + \varepsilon_{i2}, \\ &\vdots \\ Y_{iT} &= W'_{it}\gamma_i + \varepsilon_{iT}. \end{aligned}$$

The key observation of Chamberlain (1992) and Arellano and Bonhomme (2012) is that, because γ_i is a fixed coefficient within the time series of unit i, OLS can be used to estimate γ_i by using the time series data. First, I stack the time series equations in vector form, as follows:

$$Y_i = W_i\gamma_i + \varepsilon_i,$$

where $Y_i \equiv (Y_{i1}, \ldots, Y_{iT})'$, $W_i \equiv (W_{i1}, \ldots, W_{iT})'$, and $\varepsilon_i = (\varepsilon_{i1,\ldots,}\varepsilon_{iT})$. Assume that $W_i' W_i$ is full rank, which holds if and only if the columns of W_i are linearly independent. The OLS estimator of γ_i from unit i's time series is given by

$$\hat{\gamma}_i^{OLS} = (W_i' W_i)^{-1} W_i' Y_i.$$

If $W_i' W_i$ is full rank with probability 1, then

$$\mathbb{E}(\hat{\gamma}_i^{OLS}) = \mathbb{E}((W_i' W_i)^{-1} W_i' Y_i) = \mathbb{E}(\gamma_i + (W_i' W_i)^{-1} W_i' \varepsilon_i) = \mathbb{E}(\gamma_i),$$

where $\mathbb{E}((W_i' W_i)^{-1} W_i' \varepsilon_i) = 0$ by (3). The result implies that $\mathbb{E}(\gamma_i)$ can be estimated by using the following procedure:

- For each i, run the OLS on their time series to obtain $\hat{\gamma}_i^{OLS}$.

- Estimate $\mathbb{E}(\gamma_i)$ by averaging the OLS estimators, as follows:

$$\widehat{\mathbb{E}(\gamma_i)} \equiv \frac{1}{N}\sum_{i=1}^{N}\hat{\gamma}_i^{OLS} = \frac{1}{N}\sum_{i=1}^{N}(W_i' W_i)^{-1} W_i' Y_i .$$

To express it precisely, $\theta = \mathbb{E}(\gamma_i)$ satisfies the following moment condition:

$$\mathbb{E}\left((W_i' W_i)^{-1} W_i' (Y_i - W_i\theta)\right) = 0 .$$

Second, I use the generalized method of moments to estimate θ and obtain the standard error of the estimator. Refer to Arellano and Bond (1991) for details on efficient moment conditions that involve generalized least squares, instead of OLS.

A key assumption in this approach is that $W_i' W_i$ is full rank for every i. The requirement corresponds to the condition in the standard fixed-effects models in (1), that is, for the within estimator to be consistent, within variation in W_i must exist, on average, that is, $\mathbb{E}(W_i' W_i)$ must be full rank. However, the requirement is strong in random coefficient models, because γ_i is unit specific, and its distribution is unrestricted; thus, the identification of $\mathbb{E}(\gamma_i)$ will require information on every γ_i, which can be obtained only when the individual W_i has within variation for each unit. By contrast, in standard fixed-effects models, within variation is adequate, on average, because β is a fixed parameter, and information across the units can be pooled to estimate the single β.

A key intuition behind the result of $\mathbb{E}(\hat{\gamma}_i^{OLS}) = \mathbb{E}(\gamma_i)$ is that the OLS estimator $\hat{\gamma}_i^{OLS}$ is unbiased when W_i is strictly exogenous.

$$\mathbb{E}(\hat{\gamma}_i^{OLS} \mid \gamma_i, W_i) = \mathbb{E}\left((W_i' W_i)^{-1} W_i' (W_i\gamma_i + \varepsilon_i) \mid \gamma_i, W_i\right) = \gamma_i .$$

Given that T is fixed and possibly small, there is no guarantee that $\hat{\gamma}_i^{OLS}$, that is, the OLS estimator of γ_i based on T observations, is a precise estimate of γ_i. Nevertheless, $\hat{\gamma}_i^{OLS}$ is an unbiased estimate of γ_i. Hence, when $\hat{\gamma}_i^{OLS}$ is averaged across i, the unbiased error terms will average to 0, thereby making the average of $\hat{}^{OLS}$ a precise estimate of $\mathbb{E}(\gamma_i)$. Specifically, $\hat{\gamma}_i^{OLS}$ is a consistent estimator of $\mathbb{E}(\gamma_i)$ as N tends to infinity, despite each $\hat{\gamma}_i^{OLS}$ not being consistent for its corresponding γ_i.

Next, I discuss the identification of $\mathbb{E}(\gamma_i)$ when W_{it} is sequentially exogenous, as defined in (4). The model was analyzed by Chamberlain (1993), and the work was later published by Chamberlain (2022) and Lee (2022). Unfortunately, the studies showed that the consistent estimation of $\mathbb{E}(\gamma_i)$ is not possible under the assumption. The key intuition behind the result is that $\hat{\gamma}_i^{OLS}$ will no longer be unbiased when W_{it} is sequentially exogenous.

To illustrate the concept, consider a case where $W_{it} = (1, Y_{i,t-1})'$, $\gamma_i = (\alpha_i, \beta_i')'$, and $T = 2$.

$$Y_{i1} = \alpha_i + \beta_i Y_{i0} + \varepsilon_{i1},$$
$$Y_{i2} = \alpha_i + \beta_i Y_{i1} + \varepsilon_{i2},$$

where $\mathbb{E}(\varepsilon_{i1} \mid \alpha_i, \beta_i, Y_{i0}) = 0$, and $\mathbb{E}(\varepsilon_{i2} \mid \alpha_i, \beta_i, Y_{i0}, Y_{i1}) = 0$. The OLS estimator of β_i from the two-period time series is given by the ratio of the differences in Y_{it}.

$$\hat{\beta}_i^{OLS} = \frac{Y_{i2} - Y_{i1}}{Y_{i1} - Y_{i0}}.$$

The above equation is not an unbiased estimator of β_i, because

$$\hat{\beta}_i^{OLS} = \beta_i + \frac{\varepsilon_{i2} - \varepsilon_{i1}}{Y_{i1} - Y_{i0}},$$

where the second term has a nonzero expectation, because ε_{i1} and Y_{i1} are correlated. Thus, the average of $\hat{\beta}_i^{OLS}$ may not be a consistent estimator of $\mathbb{E}(\gamma_i)$, because the biased error term will not necessarily average to 0.

Lee (2022) showed that no unbiased estimator of β_i exists in this case, and $\mathbb{E}(\beta_i)$ can be identified if and only if an unbiased estimator of β_i from the time series exists. The results imply that $\mathbb{E}(\beta_i)$ is not identified, meaning that a consistent estimation for $\mathbb{E}(\beta_i)$ is not possible. This finding complements and strengthens the work of Chamberlain (1993, 2022), who showed that $\mathbb{E}(\beta_i)$ will not be identified when Y_{it} is discrete.

Lee (2022) showed that, for the general random coefficient model (2) under sequential exogeneity (4), the consistent estimation of the lower and upper bounds of $\mathbb{E}(\gamma_i)$ would be possible, which shows that

$\mathbb{E}(\gamma_i)$ is partially identified. Subsequently, I discuss the derivation of the bounds that can be estimated consistently. Consider the following moment conditions implied by (4):

$$\begin{aligned} &\mathbb{E}(\gamma_i' W_{it}(Y_{it} - W_{it}'\gamma_i)) = 0, \\ &\mathbb{E}(W_{it}(Y_{it} - W_{it}'\gamma_i)) = 0, \end{aligned} \tag{5}$$

I consider the identification of an entry in the γ_i vector. Let e be a vector that will select an entry for γ_i. For example, e = (1, 0,...,0) will select the first entry, and e = (0, 0,...,1) will select the last entry. Then, I focus on the identification of $\theta_e \equiv \mathbb{E}(e'\gamma_i)$.

For the fixed constants λ and μ, I consider the following function:

$$\mathcal{L}(\lambda, \mu, \gamma_i, W_i, Y_i) = e'\gamma_i + \lambda\sum_{t=1}^{T}\gamma_i' W_{it}(Y_{it} - W_{it}'\gamma_i) + \mu'\sum_{t=1}^{T} W_{it}(Y_{it} - W_{it}'\gamma_i).$$

Function $\mathcal{L}$ can be interpreted as the "Lagrange function," because it is the sum of the objective term $e'\gamma_i$ and the moment functions in (5), with Lagrange multipliers λ and μ.

The Lagrange function has two key properties. First, its expectation equals the parameter of interest $\mathbb{E}(e'\gamma_i)$, as follows:

$$\begin{aligned} \mathbb{E}(\mathcal{L}) = \mathbb{E}(e'\gamma_i) + \lambda\sum_{t=1}^{T}\mathbb{E}\left(\gamma_i' W_{it}(Y_{it} - W_{it}'\gamma_i)\right) + \\ \mu'\sum_{t=1}^{T}\mathbb{E}\left(W_{it}(Y_{it} - W_{it}'\gamma_i)\right) = \mathbb{E}(e'\gamma_i), \end{aligned}$$

where the last equality follows (5). Second, for each fixed (λ, μ, W_i, Y_i), $\mathcal{L}$ is a quadratic polynomial in γ_i.

$$\begin{aligned} \mathcal{L} = e'\gamma_i + \lambda\gamma_i'\sum_{t=1}^{T} W_{it}Y_{it} + \gamma_i'\left(-\lambda\sum_{t=1}^{T} W_{it}W_{it}'\right)\gamma_i + \\ \mu'\sum_{t=1}^{T} W_{it}Y_{it} - \mu'\sum_{t=1}^{T} W_{it}W_{it}'\gamma_i, \end{aligned}$$

with the leading coefficient matrix $-\lambda\sum_{t=1}^{T} W_{it}W_{it}' = -\lambda W_i' W_i$. Suppose that $W_i' W_i$ is full rank for every i, as required for the identification of $\mathbb{E}(\gamma_i)$ when W_{it} is strictly exogenous. Then, for $\lambda < 0$, the leading coefficient matrix $-\lambda W_i' W_i$ is positive definite, in which case, $\mathcal{L}$ has a finite minimum with respect to γ_i. Then, it follows that

$$\mathbb{E}(e'\gamma_i) = \mathbb{E}(\mathcal{L}(\lambda, \mu, \gamma_i, W_i, Y_i)) \geq \mathbb{E}\left(\min_{\gamma_i} \mathcal{L}(\lambda, \mu, \gamma_i, W_i, Y_i)\right),$$

where $\min_{\gamma_i} \mathcal{L}(\lambda, \mu, \gamma_i, W_i, Y_i)$ is finite for every i and therefore well defined. This calculation shows that $\mathbb{E}\left(\min_{\gamma_i} \mathcal{L}\right)$ is a lower bound of $\mathbb{E}(e'\gamma_i)$. Furthermore, because $\mathbb{E}\left(\min_{\gamma_i} \mathcal{L}\right)$ is a lower bound of all $\lambda < 0$ and μ, it follows that

$$\mathbb{E}(e'\gamma_i) \geq \max_{\lambda < 0,\ \mu} \mathbb{E}\left(\min_{\gamma_i} \mathcal{L}\right).$$

Subsequently, I derive the expression of the lower bound. For brevity of notation, let $\mathcal{W}_i = \sum_{t=1}^{T} W_{it} W_{it}'$ and $\mathcal{Y}_i = \sum_{t=1}^{T} W_{it} Y_{it}$. Then, $\mathcal{L}$ can be written as

$$\mathcal{L} = e'\gamma_i + \lambda \gamma_i' \mathcal{Y}_i + \gamma_i'(-\lambda \mathcal{W}_i)\gamma_i + \mu'\mathcal{Y}_i - \mu'\mathcal{W}_i \gamma_i .$$

I attain the minimum of $\mathcal{L}$ with respect to γ_i as the solution to the first-order condition.

$$\frac{\partial \mathcal{L}}{\partial \gamma_i} = e + \lambda \mathcal{Y}_i - 2\lambda \mathcal{W}_i \gamma_i - \mu'\mathcal{W}_i = 0,$$

which yields

$$\gamma_i^* = \frac{1}{2\lambda} \mathcal{W}_i^{-1}(e + \lambda \mathcal{Y}_i - \mu'\mathcal{W}_i).$$

The substitution of the equation into $\mathcal{L}$ yields

$$\min_{\gamma_i} \mathcal{L} = \frac{1}{4\lambda}(e + \lambda \mathcal{Y}_i - \mu'\mathcal{W}_i)' \mathcal{W}_i^{-1} (e + \lambda \mathcal{Y}_i - \mu'\mathcal{W}_i) + \mu'\mathcal{Y}_i.$$

Then, to compute the lower bound $\max_{\lambda < 0,\ \mu} \mathbb{E}\left(\min_{\gamma_i} \mathcal{L}\right)$, I maximize the expectation of the above expression with respect to $\lambda < 0$ and μ. After calculating the first-order conditions and substituting them back into the expression, I obtain the expression of the lower bound as

$$\frac{1}{2}e'\mathbb{E}\,(\hat{\gamma}_i^{OLS}) + \frac{1}{2}\hat{\gamma}_0 - \sqrt{(e'\mathbb{E}\,(\mathcal{W}_i^{-1})'e - e'\mathbb{E}\,(\mathcal{W}_i)^{-1}e)\left(\mathbb{E}\,(Y_i'\,\mathcal{W}_i^{-1}\,Y_i) - \mathbb{E}\,(Y_i)'\,\mathbb{E}\,(\mathcal{W}_i)^{-1}\,\mathbb{E}\,(Y_i)\right)},$$

where

$$\hat{\gamma}_i^{OLS} = \mathcal{W}_i^{-1}\mathcal{Y}_i, \quad \hat{\gamma}_0 = \mathbb{E}\,(\mathcal{W}_i)^{-1}\mathbb{E}\,(\mathcal{Y}_i).$$

By applying the same approach to $\lambda > 0$, I obtain the upper bound of $\mathbb{E}\,(e'\gamma_i)$ as

$$\frac{1}{2}e'\mathbb{E}\,(\hat{\gamma}_i^{OLS}) + \frac{1}{2}\hat{\gamma}_0 + \sqrt{(e'\mathbb{E}\,(\mathcal{W}_i^{-1})'e + e'\mathbb{E}\,(\mathcal{W}_i)^{-1}e)\left(\mathbb{E}\,(Y_i'\,\mathcal{W}_i^{-1}\,Y_i) - \mathbb{E}\,(Y_i)'\,\mathbb{E}\,(\mathcal{W}_i)^{-1}\,\mathbb{E}\,(Y_i)\right)}.$$

Then, I can compute the confidence interval of the bounds based on the result. Refer to Lee (2022) for details on the calculation of the confidence interval.

Lee (2022) showed that the lower and upper bounds are the sharp (*i.e.*, largest) bounds of $\mathbb{E}\,(e'\gamma_i)$ under the following moment conditions:

$$\sum_{t=1}^{T}\mathbb{E}\left(\gamma_i'\mathcal{W}_{it}(Y_{it} - \mathcal{W}_{it}'\gamma_i)\right) = 0,$$

$$\sum_{t=1}^{T}\mathbb{E}\left(Z_{it}(Y_{it} - \mathcal{W}_{it}'\gamma_i)\right) = 0,$$

which implies that the bounds are outer bounds under (4).

III. DID models

In this section, I discuss the identification and estimation of TWFE models with heterogeneous coefficients. Traditionally, TWFE models have provided the motivation for DID analyses. However, recent advancements in the DID literature (*e.g.*, Chaisemartin and

D'Haultfoeuille, 2020; Callaway, Goodman-Bacon, and Sant'Anna, 2024) emphasize that standard TWFE regressions are not consistent when the treatment effects are heterogeneous.

A standard TWFE regression specification for analyzing the treatment effect is written as

$$Y_{it} = \alpha_i + \delta_t + \beta D_{it} + \varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T, \tag{6}$$

where Y_{it} is the outcome variable, D_{it} is the binary treatment variable, α_i and δ_t are the unit and time fixed effects, and ε_{it} is an idiosyncratic error term. In DID models, D_{it} is typically assumed to be strictly exogenous, as follows:

$$\mathbb{E}(\varepsilon_{it} \mid D_{i1}, \dots, D_{iT}) = 0.$$

Similar to the one-way fixed-effects models, I consider a case of short panels, where N tends to infinity, but T is fixed in the asymptotics.

Recent advancements in the DID literature emphasize that, when the treatment effect is heterogeneous across the units and time periods, that is,

$$Y_{it} = \alpha_i + \delta_t + \beta_{it} D_{it} + \varepsilon_{it}, \tag{7}$$

coefficient β in (6) will not necessarily be equal to the expectation of β_{it} in (7), except in the canonical two-period DID model. Specifically, in the canonical model, where $T = 2$, $D_{i1} = 0$, and $D_{i2} \in \{0, 1\}$, coefficient β in (6) and β_{it} in (7) satisfy

$$\beta = \mathbb{E}(\beta_{it} \mid D_{i2} = 1),$$

where the right-hand side is referred to as the average treatment effect on the treated (ATT). However, when $T > 2$, β will not necessarily be equal to the ATT.

The literature proposed various procedures for the consistent estimation of the ATT. Next, I discuss the proposed solutions for DID models with binary treatment ($D_{it} \in \{0, 1\}$) and irreversible treatment ($D_{is} = 1$ implies $D_{it} = 1$ for $t > s$). I define

$$G_i \equiv \min \{t \mid D_{it} = 1\}$$

as the first period, when unit i enters the treatment, which is commonly referred to as "groups" in the literature. If $D_{it} = 0$ for all t, then G_i is set to ∞. It is typically assumed that $D_{i1} = 0$, which means that all the units are untreated at $t = 1$; thus, $G_i \geq 2$ for all i. Let $q \equiv \min G_i$ be the first period, when any of the units can enter the treatment, which is referred to as the first posttr eatment period in the literature.

The literature defined the parameter of interest as

$$ATT\,(g,\ t) = \mathbb{E}\,(\beta_{it} \mid G_i = g).$$

The equation represents the average treatment effect at time t for the units that entered the treatment at time g. Then, I define the aggregate ATT by averaging $ATT\,(g,\ t)$ across the different $(g,\ t)$. For example, the ATT across all the groups and periods is defined by

$$\sum_{g=q}^{T} \sum_{t=g}^{T} w_{gt} ATT\,(g,\ t),$$

where $w_{gt} \geq 0$ represents the weight on $ATT\,(g,t)$ such that $\sum_{g=q}^{T} \sum_{t=g}^{T} w_{gt} = 1\cdot$

The literature proposed various procedures for estimating $ATT\,(g,\ t)$. I describe the key idea in this paragraph. Suppose that we are interested in estimating ATT (3, 5), which is the ATT of group 3 at time 5. If we restrict the analysis to the data subset in which $G_i \in \{3,\ \infty\}$, and $t \in \{1,\ 5\}$, the subset will become the canonical DID model. Then, as discussed previously, the TWFE regression model applied to this canonical DID setup will consistently estimate the parameter of interest.

$$Y_{it} = \alpha_i + \delta_t + \beta D_{it} + \varepsilon_{it}, \quad i : G_i \in \{3,\ \infty\}, \quad t \in \{1,\ 5\},$$

where $\beta = ATT\,(3,\ 5)$.

By generalizing the idea, Wooldridge (2021, 2023) proposed the following generalized TWFE regression model, which is consistent with a general DID setup:

$$Y_{it} = \alpha_i + \delta_t + \sum_{g=q}^{T} \sum_{s=g}^{T} \beta_{gs} 1\,(G_i = g,\ t = s) D_{it} + \varepsilon_{it},$$

where $\beta_{gs} = ATT\,(g, s)$. The regression can be implemented conveniently by using a standard statistical software and regressing Y_{it} on the unit and time fixed effects (*i.e.*, unit and time indicators) and a triple interaction term of the group indicator, time indicator, and D_{it}. The standard errors of β_{gs} can also be calculated conveniently by using a standard statistical software.

Borusyak, Jaravel, and Spiess (2024) proposed an imputation-based estimator for $ATT\,(g, t)$ as an alternative. The estimator can be implemented through the following steps:

• Run a regression of Y_{it} on the unit and time fixed effects by using the data subset in which $D_{it} = 0$, as follows:

$$Y_{it} = \alpha_i + \delta_t + \varepsilon_{it}, \qquad (i, t) : D_{it} = 0.$$

Let $\hat{\alpha}_i$ and $\hat{\delta}_t$ be the estimates from the regression.

• For each (i,t), where $D_{it} = 1$, calculate the imputed baseline outcome as

$$\hat{Y}_{it} = \hat{\alpha}_i + \hat{\delta}_t.$$

• Estimate $ATT\,(g, t)$ by using

$$\widehat{ATT\,(g,\, t)} = \frac{1}{N_g} \sum_{i\,:\,G_i\,=\,g} Y_{it} - \hat{Y}_{it},$$

where $N_g = \#\{i{:}G_i = g\}$ is the number of observations such that $G_i = g$. The calculation of the standard error of $\widehat{ATT\,(g,\, t)}$ is less straightforward than the TWFE regression method proposed by Wooldridge (2021, 2023). However, an imputation-based estimator can handle unbalanced panel data, whereas the TWFE regression method is valid only for balanced panel data. In addition, Borusyak, Jaravel, and Spiess (2024) provided a statistical package to facilitate the implementation of their approach.

The procedure proposed by Callaway and Sant'Anna (2021), which I do not discuss in detail, builds on the same idea that the TWFE regression estimate is consistent when restricted to two-period data. The approach is useful when researchers wish to incorporate additional covariates into the DID models. The addition of covariates as additive terms in the TWFE regression method will be valid only if the additive specification is true. Callaway and Sant'Anna (2021) proposed an estimation method that will allow the covariates to enter nonparametrically in terms of treatment effect heterogeneity. The authors also provided a statistical package to facilitate the implementation of their approach.

IV. Conclusion

In this study, I review key ideas for the identification and estimation of fixed-effects models with heterogeneous coefficients. For one-way fixed-effects models, I discuss the identification strategies under strict and sequential exogeneity. For TWFE models, I provide a review in the context of DID models and emphasize that standard TWFE regression models are inconsistent for the ATT when the treatment effects are heterogeneous. Moreover, I discuss various alternative strategies for estimating the ATT consistently. Other references or the statistical packages provided by the authors can be used to obtain detailed information on the implementation steps.

(Submitted Jan 15, 2025; accepted Jan 15, 2025)

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