Bargaining and Walrasian Equilibrium in Simple Production Economies

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We consider a class of production economies in which there are two goods, leisure and a consumption good, and there are a large number of consumers and firms, modelled as continua. We formalize a bargaining procedure in the labor market, and examine the relation between the bargaining equilibria and the Walrasian equilibria of the underlying production economy. We show that every bargaining equilibrium allocation coincides with a Walrasian equilibrium allocation in the underlying economy. So, a Gale-type result is obtained for our production economies.

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I. Introduction

A line of study showed that sequential bargaining could provide a non-cooperative foundation for Walrasian equilibria in exchange economies: every equilibrium allocation of a bargaining game in which trades are determined through a sequential bargaining process is a Walrasian equilibrium allocation in the underlying economy. In this paper, we are concerned with a class of production economies, we formalize a bargaining procedure and clarify the relation between the bargaining equilibria and the Walrasian equilibria in the underlying production economy.

The study of the relation between the two types of equilibria

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started in Rubinstein and Wolinsky (1985) and Shaked and Sutton (1984). Rubinstein and Wolinsky formalized a bargaining procedure between buyers and sellers of an indivisible good and showed that the subgame perfect equilibrium allocation of their bargaining game was generally inconsistent with any Walrasian equilibrium allocation in the underlying economy. Similar inconsistency was obtained by Shaked and Sutton in a labor market with a single employer and many workers who bargain about the wage for one unit of indivisible labor.

In a series of papers, Gale (1984, 1986a, 1986c) formalized a sequential bargaining procedure in a "frictionless" market and showed that every subgame perfect equilibrium allocation of his bargaining game, in contrast with the result of Rubinstein and Wolinsky, was consistent with a Walrasian equilibrium allocation in the underlying economy. Conversely, Gale (1986b) showed that any Walrasian equilibrium in his exchange economy could be supported by a subgame perfect equilibrium. Gale (1986c) and McLennan and Sonnenschein (1991) obtained the same result as Gale (1986a)'s without some of the restrictive assumptions in Gale (1986a).

Their results about the relation between bargaining equilibria and Walrasian equilibria are remarkable. But, they pertain only to exchange economies. The issue of sequential bargaining in production economies remains untackled: a sequential bargaining game for production economies was not formulated in the previous literature, and therefore the relation between the two types of equilibria in production economies remains unexplored. Also, since Shaked and Sutton (1984)'s result pertains only to a labor market in which there are a single employer and many workers and labor is indivisible, the relation remains unexplored in the case of a labor market in which there are many employers and workers and labor is divisible.

The object of this paper is to examine the relation in the case of production economies as well as in the case of the labor market. To the purpose, we specifically consider a class of production economies in which there are two goods, leisure and a consumption good, and there are a large number of consumers and firms, modelled as continua. We formalize a bargaining procedure in the labor market and examine the relation between the bargaining equilibria and the Walrasian equilibria of the underlying production economy. Our main result is as follows: *Every subgame perfect* *eqilibrium allocation of the bargaining game coincides with a Walrasian equilibrium allocation in the underlying economy.*

An earlier version of this paper was an attempt at extending Gale (1986a)'s model to production economies. But, Gale's result is based on the assumption of "widely dispersed characteristics"[1] and the property is obtained by requiring that at each time, new entrants come in with widely dispersed characteristics. However, in a production economy, firms are described in terms of production technologies, not endowments. So, it would be inappropriate to mechanically extend Gale (1986a)'s model to a production economy. In the later literature (Gale (1986c), McLennan and Sonnenschein (1991)), the assumption was dropped. Instead, Gale imposed a uniform bound on the curvatures of agents' indifference surfaces. McLennan and Sonnenschein criticized the assumption as precluding many usual preferences.[2] Instead, McLennan and Sonnenschein (1991) showed that at least in their model, the assumption could be replaced by a topological restriction on the form of the equilibria of their bargaining game.

In the bargaining procedure formulated in this paper, consumers are paired with firms and they bargain over the set of net trades which make both of them better off. In a subgame perfect equilibrium of the game, the sets of preferred trades of consumers and firms that are "ready to leave the market" have a common tangent line through the origin whose normal can be interpreted as a "pseudo equilibrium price vector".

Now, consider the intersection of the sets of preferred trades of the firms that are ready to leave the market. Since for each such firm, its set of preferred trades is generated by its production technology, and since the number of different production technologies in the economy is assumed to be finite, there are at most a finite number of different sets of preferred trades of firms supported by the tangent line identified above. Thus, given a trade whose terms are less favorable to a consumer than the pseudo-price vector, there exists a trade in the intersection whose terms are more favorable to him than the given trade. Thus, we can show

[1]The assumption requires that at each time, for any open set in $\mathbb{R}^l_+$, the measure of agents whose current net endowments are in the open set is positive.

[2]For details, see McLennan and Sonnenschein (1991) p. 1418.

that for each consumer, rejecting a proposed trade whose terms are less favorable than the pseudo-price vector is a credible threat.

In the case of exchange economies, a pseudo equilibrium price vector is also determined by the tangent line through the origin to the sets of preferred trades of consumers that are ready to leave the market. But, there can be an infinite number of different sets of preferred trades that are supported by the tangent line even though all consumers have the same preferences. Thus, in the intersection of the sets of preferred trades, there may not exist a trade whose terms are sufficiently close to the pseudo-price vector unless the curvatures of the indifference curves are uniformly bounded. Therefore, given a trade less favorable to a consumer than the pseudo-price vector, there may not exist a trade in the intersection whose terms are more favorable to the consumer than the given trade.

From the above informal arguments, we discover that bargaining between a consumer and a firm instead of two consumers enables us to obtain the consistency result without imposing the restrictive assumptions that have been used in the previous literature on exchange economies.

This paper is organized as follows. Section II presents a class of production economies and related definitions and formulates our sequential bargaining game. Section III formalizes the main result and gives a proof. Section IV concludes.

II. Preliminaries

A. Simple Production Economy and Warlasian Equilibrium

We are concerned with a class of production economies in which there are two divisible goods, leisure, indexed by 1 and a consumption good, indexed by 2 and there are a large number of consumers (or workers) and firms (or employers), modelled as continua.

We assume that every consumer has the same consumption set $[0,1]\times\mathbb{R}_+$ and is endowed with 1 unit of leisure and 0 unit of the consumption good. Therefore, consumers are characterized only by their preferences defined on the consumption set. We assume that preferences are represented by von Neumann-Morgenstern utility functions.

Firms employ consumers to produce the consumption good and pay some amount of the good they produced to consumers as wage. Following Hicks (1946), we define a firm as a single entrepreneur that produces the consumption good by means of his production technology, and consumes the *surplus of his production* defined as the amount of the consumption good remaining after paying for the wage bill. Without loss of generality, we assume all entrepreneurs to have the same von Neumann-Morgenstern utility functions w defined on the set of non-negative real numbers. Thus, firms are characterized only by their production technologies which are represented by production functions.

The set of consumers is represented by an atomless probability space $(A, \boldsymbol{A}, \nu^A)$, where A is the set of "names" of consumers, $\boldsymbol{A}$ is a σ-algebra of A and ν^A is a probability measure defined on $\boldsymbol{A}$. The set of firms is represented by an atomless probability space $(B, \boldsymbol{B}, \nu^B)$ which is defined symmetrically. Let $\boldsymbol{U}$ be the family of admissible (von Neumann-Morgenstern) utility functions $u:[0,1]\times\mathbb{R}_+\mapsto\mathbb{R}$, and $\boldsymbol{F}$ be the family of admissible production functions $f:\mathbb{R}_-\mapsto\mathbb{R}_+$. We impose the following conditions on $\boldsymbol{U}$, $\boldsymbol{F}$ and the entrepreneurs' common utility function $w:\mathbb{R}_+\mapsto\mathbb{R}$.

***Assumption* 1.**

(C) For all $u\in\boldsymbol{U}$,

1. u is continuous, bounded above, strictly increasing, continuously differentiable and strictly concave,
2. $u(x)=0$ if either $x_1=0$ or $x_2=0$

(P) For all $f\in\boldsymbol{F}$,

1. f is continuous, bounded above, strictly decreasing, continuously differentiable and strictly concave,
2. $f(0)=0$ and $df(l)/dl\to-\infty$ as $l\to0^-$.

(E) 1. w is continuous, bounded above, strictly increasing and strictly concave
   2. $w(0)=0$.[3]

A *simple production economy* ε is a pair of measurable functions[4]

[3](C)2 and (E)2 are for normalization.

[4]The sets $\boldsymbol{U}$ and $\boldsymbol{F}$ are assumed to be endowed with the uniform convergence topology.

$\varepsilon^A:(A, \boldsymbol{A}, \nu^A)\mapsto\boldsymbol{U}$ and $\varepsilon^B:(B, \boldsymbol{B}, \nu^B)\mapsto\boldsymbol{F}$.

For each consumer $a\in A$, $\varepsilon^A(a)$ is his utility function and for each firm $b\in B$, $\varepsilon^B(b)$ is its production function. We impose the following conditions on ε^A and ε^B.

***Assumption* 2.** (finite supports) $\text{supp}(\nu^A\circ(\varepsilon^A)^{-1})$ and $\text{supp}(\nu^B\circ(\varepsilon^B)^{-1})$ are finite.

This assumption is a version of Gale (1986c)'s Assumption 1 for the present context of production economies. It implies that there are only finite types of agents except for a set of agents of measure zero. Thus, we think of the economy as an infinite replica of an economy with a finite number of agents.

An allocation for the economy ε is a pair of integrable functions $\phi:A\mapsto\mathbb{R}^2$ and $\psi:B\mapsto\mathbb{R}^2$ such that for ν^A—*a.e.* a in A, $\phi(a)\in[0,1]\times\mathbb{R}_+$ and for ν^B—*a.e.* b in B, $\psi(b)\in[0,1]\mathbb{R}_-\times\mathbb{R}_+$. The function ϕ assigns to almost every consumer a bundle consisting of some amount of leisure and some amount of the consumption good in the consumption set, and the function ψ assigns to almost every firm a bundle consisting of some amount of labor and some amount of the consumption good.

For all b in B, let $f^b=\varepsilon^B(b)$. Then, an *attainable allocation for the economy* ε is an allocation (ϕ,ψ) such that

$$\text{(i)}\ \int_A\phi_1 d\nu^A+\int_B(-\psi_1)d\nu^B\le\int_A d\nu^A$$

$$\text{(ii)}\ \int_A\phi_2 d\nu^A+\int_B\psi_2 d\nu^B\le\int_B f^b\circ\psi_1 d\nu^B$$

Condition (i) requires that the sum of the mean consumption of leisure and the mean labor input be at most equal to the mean endowment of leisure. Condition (ii) requires that the sum of consumers' mean consumption of the consumption good and entrepreneurs' mean consumption be at most equal to the mean production of the good.

For all a in A, let $u^a=\varepsilon^A(a)$ and for any given $p\in\mathbb{R}_+$, for all $x\in\mathbb{R}^2$, let

$$B(p,x)\equiv\{x'\in[0,1]\times\mathbb{R}_+\,|\,(p,1)\cdot x'\le(p,1)\cdot x\}$$

Definition. A *Walrasian equilibrium of the economy* ε is an attainable allocation (ϕ, ψ) and a price $p \in \mathbb{R}_+$ of leisure such that

(i) for $\nu^A - a.e.$ $a \in A$, $\phi(a) = \arg\max u^a(x)$ on $B(p,(1,0))$,
(ii) for $\nu^B - a.e.$ $b \in B$, $\phi_1(a) = \arg\max w(f^b(l) + pl)$ and $\phi_2(b) = f^b(\phi_1(b)) + p\,\phi_1(b)$.

B. The Bargaining Model

We formalize a sequential bargaining procedure for the economy ε, extending Gale (1986a,c)'s earlier work. Bargaining takes place in the labor market. Time is divided into discrete intervals indexed by non-negative integers, and the whole time horizon is partitioned as follows:

$T_0 = 3\mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers.
$T_1 = T_0 + \{1\}$
$T_2 = T_0 + \{2\}$.

The set of consumers in the market is represented by a measure m^A defined on the Borel sets of $\boldsymbol{U}$ such that

$$m^A = \nu^A \circ (\varepsilon^A)^{-1}$$

and the set of firms in the market is represented by a measure m^B defind on the Borel sets of $\boldsymbol{F}$ such that

$$m^B = \nu^B \circ (\varepsilon^B)^{-1}.$$

All consumers and firms enter the market at time 0. At each time $t \in T_0$, a pairing between consumers and firms occurs and for each matched pair of a consumer and a firm, one of them is randomly chosen as the proposer. At time $t+1 \in T_1$, each proposer makes a trade of labor and the consumption good. At time $t+2 \in T_2$, each responder accepts, or rejects the proposal and stays in the market, or rejects the proposal and leaves the market. Even if a pair of a consumer and a firm agree on a trade, they do not deliver real goods immediately. Instead, each agent involved in the trade writes to his partner a claim[5] for the good that he should deliver as specified in the accepted proposal. Agents who decided to stay in the market at time $t+2$ are paired with other partners after time $t+2$ and the same procedure is repeated.

We make the following assumption on claims that are exchanged in the market.

***Assumption* 3.** Claims are transferable. Each agent can require fulfillment of a claim he holds only after he leaves the market, and each agent can fulfil a claim that he is liable for only after he leaves the market.

If claims are not transferable, consumers can not increase leisure in the bargaining procedure. So, the transferability assumption removes such a "friction" in the underlying economy. If an agent stays in the market forever, then he can never fulfil the claims that he is liable for. If an agent who never fulfills claims he is liable for is penalized.

***Assumption* 4.** The utility of being penalized is $-\infty$.

Although consumers are initially distinguished by their utility functions, after time 0, they are distinguished by three data, their utility functions, their current net endowment of leisure and their current net amount of claims for the consumption good that they hold.[6] We call the three data their *current characteristics* and the pair of the second and the third data their *consumption plans*. Similarly, after time 0, firms are distinguished by three data, their production functions, their current net amounts of claims for labor and their current net amounts claims for the consumption good that they are liable for.[7] We call the three data their *current characteristics* and the pair of the second and the third data their current *employment-wage plans*.

[5]A *claim* is a contract for the transfer of a good from one agent to another.

[6]Let $z_1, z_2, \cdots, z_n \in \mathbb{R}^2$ be the sequence of trades that a consumer has made. Then, his current net endowment of leisure is equal to $1+\sum_1^n z_{k1}$ and his current net amount of claims for the consumption good is equal to $\sum_1^n z_{k2}$.

[7]Let $z_1, z_2, \cdots, z_m \in \mathbb{R}^2$ be the sequence of trades that a firm has made. Then, its current net amount of claims for labor is equal to $\sum_1^n z_{k1}$ and its current net amount of claims for the consumption good that the firm is liable for is equal to $1+\sum_1^n z_{k2}$.

Now, we formalize the set of agents' actions. At time $t \in T_1$, each proposer makes a proposal of a trade of labor and the consumption good. So, the *set of proposers' actions* is $\mathbb{R}^2$.

***Remark* 1**
Gale (1986a) required that every trade proposed by an agent be feasible in the following sense: if x is his current consumption plan and z is the trade he proposes, then $x+z$ must be in his consumption set. But, we do not require such a feasibility condition. Instead, we follow the "short sales" assumption of McLennan and Sonnenschein (1991), and allow a consumer's current consumption plan to be outside of his consumption set, and a firm's current employment-wage plan to be such that its current total wage bill is greater than the amount of the consumption good that the firm produces with its current employment. Since, in contrast to what is the case in Gale's model, agents exchange claims for goods instead of exchanging real goods, such short sales are always possible by issuing claims.

***Remark* 2**
We also allow agents to propose a trade z in which $z_1 > 0$. In other words, a consumer is allowed to receive from a firm a claim for other consumers' labor. The firm transfers to the consumer some of the claims for labor that it has been holding. The consumer may, in turn, send the claims to firms that had been holding claims for his own labor, while nullifying the same amount of claims for his own labor, or he may use the claims for future trades.

At time $t \in T_2$, each responder can (i) accept the proposal and stay in the market, (ii) reject the proposal and stay in the market, or (iii) if the responder's current consumption or employment-wage plan is feasible,[8] he can reject the proposal and leave the market. So, the *set of responders' actions* is $\{Y,N,L\}$, where Y denotes action (i), N denotes action (ii), L denotes action (iii).

At each time t, *a history of an agent at time t*, is a finite sequence

[8]That is, the current consumption plan x is in $[0,1] \times \mathbb{R}_+$ or the current employment-wage plan y satisfies $y_2 \leq f(y_1)$, where f is its production function.

$$h_t = \{c_0, e_0, e_1, \cdots, e_{t-1}\},$$

where c_0 is his *initial characteristic* (*i.e.*, a utility function or a production function) and for each $s = 0, 1, \cdots, t-1$, e_s is his *experience* at time s defined as follows: for each $s \in T_0$, if the agent is unmatched at time s, then $e_s = e_{s+1} = e_{s+2} = \varnothing$. For each $s \in T_0$, if the agent is matched at time s, then e_s is composed of his own and his partner's current characteristics and the choice of proposer at time s, e_{s+1} is the proposer's action at time $s+1$ and e_{s+2} is responder's action at time $s+2$.

For each t, let H_t be the set of admissible histories at time t and for each $t \in T_1 \cup T_2$, let $\vec{H_t}$ be the set of admissible histories of each agent that makes a move at time t. For each $t \in T_0$, for each $h \in H_t$, let $\gamma_t(h)$ denote the current characteristic of an agent distinguished by the history h at time t.[9]

Now, we are ready to define a strategy profile. As in Gale (1986a), since agents in the market are distinguished not by "names" but by histories, we can define a strategy profile for all agents by a sequence of functions as follows.

Definition. A *strategy profile* σ is a sequence of functions $\{\sigma_t\}_{t \in T_1 \cup T_2}$ such that

(i) for all $t \in T_1$, $\sigma_t : \vec{H_t} \mapsto \mathbb{R}^2$ and for all $t \in T_2$, $\sigma_t : \vec{H_t} \mapsto \{Y, N, L\}$
(ii) for all $t \in T_2$, for all $h \in \vec{H_t}$, if $\sigma_t(h) = L$, then $\gamma_t(h)$ is feasible,
(iii) for all t, σ_t is measurable.[10]

Let Σ be the set of admissible strategy profiles.

Before describing the matching process, we impose a condition for consumers to be allowed to play the game. At each time $t \in T_0$, a consumer distinguished by a history h is *disqualified* for playing the game if $(\gamma_{t2}(h), \gamma_{t3}(h)) \in \mathbb{R}^2_-$.

***Assumption* 5.** At each time $t \in T_0$, no disqualified consumer is matched.

[9]That is, if an agent is a consumer, then $\gamma_{t1}(h)$ is his utility function and $(\gamma_{t2}(h), \gamma_{t3}(h))$ is his current consumption plan at time t and if the agent is a firm, then $\gamma_{t1}(h)$ is its production function and $(\gamma_{t2}(h), \gamma_{t3}(h))$ is its current employment-wage plan at time t.

[10]For each function σ_t the $\sigma-$algebras of the domain and the range are generated in the same way as in Gale (1986a) footnote 8.

This assumption is based on the usual norm in actual market trading that only workers who have something to trade can participate in bargaining.

Now, we describe the matching process between consumers and firms. At each time $t \in T_0$, a consumer in the market is randomly matched with a firm in the market or is unmatched and, a firm in the market is randomly matched with a consumer in the market or is unmatched. The matching satisfies the following conditions.

(i) For each $t \in T_0$, for each qualified consumer in the market at time t, the probability of being matched is $\alpha_t \in (0,1)$ and for each firm in the market at time t, the probability of being matched is $\beta_t \in (0,1)$.

(ii) For each $t \in T_0$, for each qualified consumer in the market at time t, the set of potential partners is the set of employers in the market at time t. Each qualified consumer faces the same probability distribution of potential partners at time t. Similarly, for each $t \in T_0$, for each firm in the market at time t, the set of potential partners is the set of qualified consumers in the market at time t. Each firm faces the same probability distribution of potential partners at time t.

(iii) For each member of any given matched pair, the probability of being chosen as the proposer is always 1/2.

For each t, let $H^A_t \equiv \{h_t \in H_t \mid h_{t1} \in U\}$ and $H^B_t \equiv \{h_t \in H_t \mid h_{t1} \in F\}$. Then, H^A_t is the set of admissible histories of consumers at time t and H^B_t is the set of admissible histories of firms at time t. At each time $t \in T_0$, the distributions of potential partners are represented by a pair of probability measures P^A_t on H^A_t and P^B_t on H^B_t. That is, for each measurable set $E \subseteq H^A_t$, $P^A_t(E)$ is the probability that a matched firm at time t meets a consumer whose history belongs to the set E. Similarly, for each measurable set $G \subseteq H^B_t$, $P^B_t(G)$ is the probability that a matched consumer at time t meets a firm whose history belongs to the set G.

Let Π^A be the family of admissible sequences of probability measures $\{P^A_t\}$, and Π^B be the family of admissible sequences of probability measures $\{P^B_t\}$. Then, the pair of sequences of matching probability measures are determined by the following functions:

$$\xi^A : \Sigma \mapsto \Pi^A \quad \text{and} \quad \xi^B : \Sigma \mapsto \Pi^B.$$

That is, given a strategy profile, the pair of sequences of the matching probability measures are uniquely determined and we assume that the functions ξ^A and ξ^B are parameters of the game.[11]

Before defining payoff functions, we need to note that the following facts: since an agent staying in the market forever gets $-\infty$ utility, no agent will play a strategy that prescribes staying in the market forever with a positive probability. Since each agent is paired with at most countably many partners, it follows that for each consumer, the set of agents that are holding claims for his labor or against whom he holds claims for the consumption good is also countable. Thus, all agents in the set leave the market with probability 1. So, each consumer believes that the claims for his labor and his claims for the consumption good are fulfilled with probability 1. By the same argument, each firm believes that its claims for labor and the claims for the consumption good that it will produce are fulfilled with probability 1.

Now, we specify the payoff functions. Because of the uncertainty in the matching process, given a strategy profile, the terminal consumption or employment-wage plan with which an agent leaves the market is a random variable.

Given a strategy profile σ, let $\widetilde{X}^{\sigma}$ be the associated random terminal consumption plan. Then, the expected utility of a consumer distinguished by a history h is determined by the function $U: \Sigma \times (\cup_t H_t^A) \mapsto \mathbb{R}$ defined as follows: for all $\sigma \in \Sigma$, for all $h \in \cup_t H_t^A$,

$$U(\sigma, h) = E(u(\widetilde{X}^{\sigma}) \mid h),$$

where $u = h_1$, the utility function of the consumer.

For any $f \in F$, for any terminal employment-wage plan $y \in \mathbb{R}_- \times \mathbb{R}$, the profit of a firm with the production function f is defined as

$$\pi^f(y) = f(y_1) - y_2.$$

Given a strategy profile σ, let $\widetilde{Y}^{\sigma}$ be the random terminal employment-wage plan. Then, the expected utility of a firm (that is,

[11]In Gale (1986a), the sequences of matching probability measures themselves are assumed to be parameters of the game. In terms of our model, it is equivalent to that ξ^A and ξ^B are constant functions.

entrepreneur) distinguished by a history h is determined by the function $W: \Sigma \times (\cup_t H_t^B) \mapsto \mathbb{R}$ defined as follows: for all $\sigma \in \Sigma$, for all $h \in \cup_t H_t^B$,

$$W(\sigma, h) = E(w(\pi^f(\widetilde{Y}^{\sigma})) \mid h),$$

where $f = h_1$, the production function of the firm.

Now, the bargaining game in extensive form is specified by the list

$$\Gamma = \{(U, m^A), (F, m^B), \{H_t\}, \Sigma, (\{\alpha_t\}, \{\beta_t\}, \xi^A, \xi^B), (U, W)\}.$$

For all $t \in T_1 \cup T_2$, for all $h \in \vec{H}_t$, and for all $s \in T_1 \cup T_2$, let

$$\vec{H}_s(h) \equiv \{h' \in \vec{H}_s \mid h' = (h, e_t, e_{t+1}, \cdots, e_{s-1})\}$$

and $H_s(-h) \equiv \vec{H}_s(h) \setminus \vec{H}_s(h)$.[12] Then, given a strategy profile $\sigma \in \Sigma$, given $t \in T_1 \cup T_2$, given $h \in \vec{H}_t$, let $\sigma|_h$ denote the sequence of the functions $\{\sigma_s|_{\vec{H}_s(h)}\}_{s \in T_1 \cup T_2}$ and $\sigma|_{-h}$ denote the sequence of the functions $\{\sigma_s|_{\vec{H}_s(-h)}\}_{s \in T_1 \cup T_2}$. Then $\sigma|_h$ is the continuation strategy of an agent distinguished by the history h at time t and $\sigma|_{-h}$ is the strategies of the other agents.

Definition. A *subgame perfect equilibrium* of the bargaining game Γ is a strategy profile $\sigma^* \in \Sigma$ such that

(i) $\forall h \in \cup_t \vec{H}_t^A$, $\forall \sigma \in \Sigma$, $U(\sigma^*|_h, \sigma^*|_{-h}, h) \geq U(\sigma|_h, \sigma^*|_{-h}, h)$,

(ii) $\forall h \in \cup_t \vec{H}_t^B$, $\forall \sigma \in \Sigma$, $W(\sigma^*|_h, \sigma^*|_{-h}, h) \geq W(\sigma|_h, \sigma^*|_{-h}, h)$.

III. Theorem and a Proof

We formalize the main result and give a proof. For that purpose, we introduce the notion of *competitiveness* of a strategy profile. Given a strategy profile $\sigma \in \Sigma$, given $u \in U$, let $\widetilde{X}^{\sigma}(u)$ be the random terminal consumption plan of a consumer with utility function u, and given $f \in F$, let $\widetilde{Y}^{\sigma}(f)$ be the random terminal employment-wage plan of a firm with production function f. Recall that the

[12] If $h = \vec{H}_t$, then for all $s \in T_1 \cup T_2$ with $s < t$, $\vec{H}s(h) = \varnothing$. Thus for all $s \in T_1 \cup T_2$ with $s < t$, $\vec{H}s(-h) = \vec{H}s$.

production economy underlying the game Γ is represented by a pair of measurable functions $\varepsilon^A:(A,\mathbf{A},\nu^A)\mapsto\boldsymbol{U}$ and $\varepsilon^B:(B,\mathbf{B},\nu^B)\mapsto\boldsymbol{F}$.

Definition. A strategy profile $\sigma\in\Sigma$ is competitive if there exists a Warlasian equilibrium (ϕ,ψ,p^*) of the underlying economy such that

(i) for $\nu^A-a.e.$ $a\in A$, $\widetilde{X}^{\sigma}(\varepsilon^A(a))=\phi(a)$ with probability 1,

(ii) for $\nu^B-a.e.$ $b\in B$, $\widetilde{Y}^{\sigma}(\varepsilon^B(b))=(\phi_1(b),-p^*\phi_1(b))$ with probability 1.

Under a competitive strategy, there is a Walrasian equilibrium such that almost every consumer almost surely obtains the same consumption as he would obtain at the Walrasian equilibrium and almost every firm almost surely employs the same amount of labor and obtains the same profit as it would obtain at the Walrasian equilibrium.

The following theorem is the main result of the paper.

Theorem. Every subgame perfect equilibrium of the bargaining game Γ is competitive.

The proof of the theorem relies on several preliminary results, some of which are variants of the results in Gale (1986a, c).

Let σ^* be a subgame perfect equilibrium of the game Γ. For each $t\in T_0$, define the function $V_t:H_t\mapsto\mathbb{R}$ as follows: for each $h\in H_t$,

$$V_t(h)=\begin{cases} U(\sigma^*,h) & \text{if } h\in H_t^A, \\ W(\sigma^*,h) & \text{if } h\in H_t^B. \end{cases}$$

Then, $V_t(h)$ is the equilibrium expected utility of an agent distinguished by the history h at time t. As in Gale (1986a)'s proposition 1, for each consumer, the equilibrium expected utility $V_t(h)$ depends only on his current characteristic $\gamma_t(h)$, not on all the elements in h. It is also true of firms. So, we can write $V_t(c)$ for $V_t(h)$ where $c=\gamma_t(h)$.

Let $\boldsymbol{C}^A\equiv U\times\mathbb{R}^2$ and $\boldsymbol{C}^B\equiv F\times\mathbb{R}^2$. Let $\{P_t^A\}=\xi^A(\sigma^*)$ and $\{P_t^B\}=\xi^B(\sigma^*)$. For each $t\in T_0$, for each measurable subset $\boldsymbol{C}'$ of $\boldsymbol{C}^A$, define

$$\mu_t^A(\boldsymbol{C}')=P_t^A\{h\in H_t^A\mid\gamma_t(h)\in\boldsymbol{C}'\},$$

and for each measurable subset $\mathbf{C}''$ of $\mathbf{C}^B$, define[13]

$$\mu^B_t(\mathbf{C}'') = P^B_t\{h \in H^B_t \mid \gamma_t(h) \in \mathbf{C}''\}.$$

Then, μ^A_t represents the distribution of consumers' characteristics at time t induced by the matching probability P^A_t. That is, $\mu^A_t(\mathbf{C}')$ is the probability that a matched firm meets a consumer with a characteristic in $\mathbf{C}'$ at time t. Similarly, μ^B_t represents the distribution of firms' characteristics at time t induced by the matching probability P^B_t. That is, $\mu^B_t(\mathbf{C}'')$ is the probability that a matched consumer meets a firm with a characteristic in $\mathbf{C}''$ at time t.

Now, we introduce the notion of "being ready to leave the market," following Osborne and Rubinstein (1990)'s terminology. Let

$$\mathbf{C}^A_+ \equiv U \times [0,1] \times \mathbb{R}_+.$$

For a consumer with his characteristic $(u,x) \in \mathbf{C}^A_+$ at time $t \in T_0$, suppose that $V_t(u,x) = u(x)$. Since no matter what kind of a partner he may meet, he can obtain at least the utility $u(x)$ by leaving the market at some time, $V_t(u,x) = u(x)$ implies that at any time $t' \geq t$, the probability of meeting a partner that proposes or accepts a trade z such that $V_{t'+3}(u,x+z) > u(x)$ is zero. So, $V_t(u,x) = u(x)$ implies that if the consumer is matched and chosen as responder at time $t' \geq t$, then he may leave the market at time $t'+2$. Formally, for each $t \in T_0$, a consumer with the characteristic $(u,x) \in \mathbf{C}^A_+$ at time t is ready to leave the market at time t if $V_t(u,x) = u(x)$. Similarly, let

$$\mathbf{C}^B_+ \equiv \{(f,y) \in \mathbf{C}^B \mid \pi^f(y) \geq 0\}.$$

Then, for each $t \in T_0$, a firm with the characteristic $(f,y) \in \mathbf{C}^B_+$ at time t is ready to leave the market at time t if $V_t(f,y) = w(\pi^f(y))$.

The next lemma shows that for each $t \in T_0$, there is some future time at which a positive measure of firms are ready to leave the market.

***Lemma* 1.** For each $t \in T_0$, there exists $t' \in T_0$ with $t' \geq t$ and a

[13]Since for each $t \in T_0$, the function γ_t is measurable (See Gale (1986a) p. 796), the sets $\{h \in H^A_t \mid \gamma_t(h) \in \mathbf{C}'\}$ and $\{h \in H^B_t \mid \gamma_t(h) \in \mathbf{C}''\}$ are also measurable.

measurable subset $\boldsymbol{C}'$ of $\boldsymbol{C}^B_+$ such that $\mu^B_{t'}(\boldsymbol{C}')>0$ and for each (f,y) $\in\boldsymbol{C}'$, $V_{t'}(f,y)=w(\pi^f(y))$.

Proof. See Appendix.

The next lemma is the counterpart of Lemma 1 for consumers. The proof is similar.

***Lemma* 2.** For each $t\in T_0$, there exists $t''\in T_0$ with $t''\geq t$ and a measurable subset $\boldsymbol{C}''$ of $\boldsymbol{C}^A_+$ such that $\mu^A_{t'}(\boldsymbol{C}'')>0$ and for each $(u,x)\in\boldsymbol{C}'$, $V_{t'}(u,x)=u(x)$.

For all $(u,x)\in\boldsymbol{C}^A{}_+$, let

$$Z^A(u,x)\equiv\{z\in\mathbb{R}^2\,|\,u(x+z)>u(x)\},$$

and for all $(f,y)\in\boldsymbol{C}^B{}_+$, let

$$Z^B(f,y)\equiv\{z\in\mathbb{R}^2\,|\,\pi^f(y+z)>\pi^f(y)\}.$$

Then, $Z^A(u,x)$ is the set of preferred trades for a consumer with characteristic (u,x) and $Z^B(f,y)$ is the set of preferred trades for a firm with characteristic (f,y). The next lemma, a variant of Gale (1986a)'s Proposition 3, shows that if an agent is ready to leave the market, then he has no further opportunity to make a preferred trade. We omit the proof.

***Lemma* 3.**

(i) For all $t\in T_0$, for all $(f,y)\in\boldsymbol{C}^B_+$ with $V_{t'}(f,y)=w(\pi^f(y))$, the following holds: for all $t'\in T_0$ with $t'\geq t$, for $\mu^A_{t'}-a.e.$ $(u,x)\in\boldsymbol{C}^A$, if $z\in Z^B(f,y)$, then $V_{t'+3}(u,x+z)\leq V_{t'+3}(u,x)$.

(ii) For all $t\in T_0$, for all $(u,x)\in\boldsymbol{C}^A_+$ with $V_t(u,x)=u(x)$, the following holds: for all $t''\in T_0$ with $t''\geq t$, for $\mu^B_{t'}-a.e.$ $(f,y)\in\boldsymbol{C}^B$, if $z\in Z^A(u,x)$, then $V_{t'+3}(f,y+z)\leq V_{t'+3}(f,y)$.

The next lemma shows that at some time in T_0, there is a positive measure of consumers ready to leave the market with current consumption plans in the interior of their consumption sets.

***Lemma* 4**. (Interiority) There exist $\hat{t}\in T_0$ and $\mathbf{C}\subset\hat{\boldsymbol{C}}^A_+$ such that $\mu^A_{\hat{t}}(\hat{\mathbf{C}})$ >0 and for all $(u,x)\in\hat{\mathbf{C}}$, x lies in the interior of $[0,1]\times\mathbb{R}_+$ and

$V_{\hat{t}}(u,x)=u(x)$.

Proof. See Appendix.

The next proposition shows the existence of a "pseudo equilibrium price" of leisure.

***Proposition* 1.** There exists $p\in(0,\infty)$ such that for all $t\in T_0$ for μ^B_t $-a.e.$ $(f,y)\in \boldsymbol{C}^B_+$ with $V_t(f,y)=w(\pi^f(y))$, $(p,1)$ is normal to the supporting line of $Z^B(f,y)$ through 0.

Proof. Recall that $\hat{t}$ and $\hat{\boldsymbol{C}}$ are the time and the set of characteristics of consumers, respectively referred to in Lemma 4.

***Case* 1.** $t\geq\hat{t}$.

Pick a consumer with the characteristic $(\hat{u},\hat{x})\in\hat{\boldsymbol{C}}$ at time $\hat{t}$. Let $(p,1)\equiv\boldsymbol{D}\hat{u}(\hat{x})$. Suppose to the contrary that there exist $t'\geq\hat{t}$ and $\boldsymbol{C}'\subset\boldsymbol{C}^B_+$ such that $\mu^B_t(\boldsymbol{C}')>0$ and for all $(f,y)\in\boldsymbol{C}'$, $V_{t'}(f,y)=w(\pi^f(y))$ and $(p,1)$ is not normal to the supporting line of $Z^B(f,y)$. Then, for all $(f,y)\in\boldsymbol{C}'$, there exists $z\in Z^A(\hat{u},\hat{x})$ such that $\pi^f(y+z)>\pi^f(y)$. Since $V_{t'+3}(f,y+z)\geq w(\pi^f(y+z))$, $V_{t'}(f,y)=w(\pi^f(y))$ and $V_{t'}(f,y)>V_{t'+3}(f,y)$, we obtain $V_{t'+3}(f,y+z)>V_{t'+3}(f,y)$, which contradicts Lemma 3(ii).

***Case* 2.** $t<\hat{t}$.

Since the proposition holds for Case 1, it follows from Lemma 1 and Lemma 3(ii), that for each consumer with the characteristic $(u,x)\in\hat{\boldsymbol{C}}$ at time $\hat{t}$, $(p,1)=\boldsymbol{D}u(x)$. Suppose to the contrary that there exists a firm with the characteristic $(f,y)\in\boldsymbol{C}^B_+$ at time $s<\hat{t}$ such that $V_s(f,y)=w(\pi^f(y))$ and $(p,1)$ is not normal to the supporting line of $Z^B(f,y)$ through 0. Then, for each consumer with the characteristic $(u,x)\in\hat{\boldsymbol{C}}$ at time $\hat{t}$, there exists $z\in Z^B(f,y)$ such that $(u,x+z)>u(x)$. Since $V_{\hat{t}+3}(u,x+z)\geq u(x+z)$, $V_{\hat{t}}(u,x)=u(x)$ and $V_{\hat{t}}(u,x)\geq V_{\hat{t}+3}(u,x)$, we have $V_{\hat{t}+3}(u,x+z)>V_{\hat{t}+3}(u,x)$, which contradicts Lemma 3(i).

The next lemma shows that each agent has the opportunity to make proposals to partners that are ready to leave the market as many times as he wishes. Similar results are established in Gale(1986c) and Osborne and Rubinstein (1990). (For a proof, see Osborne and Rubinstein (1990) pp. 163-4.)

***Lemma* 5**. For each $t\in T_0$, for any agent (consumer or firm) in the market at time t, the probability that at some time $s\geq t$, he is matched with a partner ready to leave the market and is chosen as proposer is 1.

The next proposition, a variant of Lemma 2 in Gale(1986c), shows that at each time, the equilibrium expected utility of a consumer is at least the maximum utility on his budget set determined by the pseudo-price. (Recall that p is the pseudo-price referred to in Proposition 1.)

***Proposition* 2.** For all $t \in T_0$, for all $(u,x) \in \boldsymbol{C}^A$, if the interior of $B(p,x)$ is non-empty, then $V_t(u,x) \geq \max u(B(p,x))$.

***Proof*.** If $u(x) = \max u(B(p,x))$, then we are done. So, we need consider only the case that $u(x) < \max u(B(p,x))$ or $x \not\ni [0,1] \times \mathbb{R}_+$. Suppose to the contrary that there exist $\bar{t} \in T_0$ and $(\bar{u},\bar{x}) \in \boldsymbol{C}^A$ such that the interior of $B(p,\bar{x})$ is non-empty and $V_{\bar{t}}(\bar{u},\bar{x}) < \max \bar{u}(B(p,\bar{x}))$. Then, there exists $x' \in [0,1] \times \mathbb{R}_+$ such that $V_{\bar{t}}(\bar{u},\bar{x}) < \bar{u}(\bar{x}) < \max \bar{u}(B(p,\bar{x}))$ and $(p,1) \cdot (x' - \bar{x}) < 0$.

***Case* 1.** $\bar{x}_1 > x'_1$.
For each $s \in T_0$ with $s \geq \bar{t}$, let $\hat{\boldsymbol{C}}_s \equiv \{(f,y) \in \boldsymbol{C}^B_+ \mid V_s(f,y) = w(\pi^f(y))\}$. Then, $\hat{\boldsymbol{C}}_s$ is the set of characteristics of firms that are ready to leave the market at time s.

Step 1. We show that there exists $z \in \mathbb{R}^2$ such that for all $s \geq \bar{t}$, for $\mu^B_s - a.e.$ $(f,y) \in \hat{\boldsymbol{C}}_s$, $z \in Z^B(f,y)$ and $nz = x' - \bar{x}$ for some natural number n.
Note that for any (f,y), (f',y') in $\boldsymbol{C}^B_+$, if $f = f'$ and $\boldsymbol{D}\pi^f(y) = \boldsymbol{D}\pi^{f'}(y')$, then $Z^B(f,y) = Z^B(f',y')$. By Proposition 1, for all $s \geq \bar{t}$, for $\mu^B_s - a.e.$ $(f,y) \in \hat{\boldsymbol{C}}_s$, $\boldsymbol{D}\pi^f(y) = -(p,1)$. And, by the "finite supports" assumption, the support of the measure m^B on the Borel sets of $\boldsymbol{F}$ is finite. Thus, there exists a finite subset Z' of $\{Z^B(f,y) \mid (f,y) \in \hat{\boldsymbol{C}}_s$ for some $s \geq \bar{t}\}$ such that for all $s \geq \bar{t}$, for $\mu^B_s - a.e.$ $(f,y) \in \hat{\boldsymbol{C}}_s$, $Z^B(f,y) \in Z'$. Thus, there exists $\tilde{z} \in \mathbb{R}^2$ such that for all for $s \geq \bar{t}$, for $\mu^B_s - a.e.$ $(f,y) \in \hat{\boldsymbol{C}}_s$, $\tilde{z} \in Z^B(f,y)$ and $x' - \bar{x} = n\tilde{z}$ for some natural number n.

Step 2. We claim that σ^* is not a subgame perfect equilibrium. Pick a history $\bar{h} \in \vec{H}^A_{\bar{t}}$ such that $\gamma_t(\bar{h}) = (\bar{u},\bar{x})$. We define a strategy profile σ' as follows: if $s \in T_1 \cup T_2$, $s \geq \bar{t}$ and $h_s = (\bar{h}, e_{\bar{t}+1}, e_{\bar{t}+2},, e_{s-1}) \in \vec{H}^A_s$, then let

$$\sigma_s'(h_s)=\begin{cases}\tilde{z} & \text{if } s\in T_1 \text{ and } \gamma_{s2}(h_s)>x_1', \\ N & \text{if } s\in T_2 \text{ and } \gamma_{s2}(h_s)>x_1', \\ (0,0) & \text{if } s\in T_1 \text{ and } \gamma_{s2}(h_s)\le x_1', \\ L & \text{if } s\in T_2 \text{ and } \gamma_{s2}(h_s)\le x_1',\end{cases}$$

For any other $(h_s)\in \overrightarrow{H}_s^A$ with $s\in T_1\cup T_1$, let $\sigma_s'(h_s)=\sigma_s^*(h_s)$. Then, under $\sigma'|_{\bar{h}}$, a consumer distinguished by the history $\bar{h}$ takes the following actions: (i) before his consumption plan reaches x', (i-1) if he is chosen as proposer, then he always proposes $\tilde{z}$, and (i-2) if he is chosen as responder, then he rejects all proposals staying in the market, and (ii) once his consumption plan reaches x', (ii-1) if he is chosen as proposer, then he always proposes (0,0), and (ii-2) if he is chosen as responder, then he leaves the market rejecting all proposals.

Suppose that a firm with characteristic $(f',y')\in \boldsymbol{C}_+^B$ is ready to leave the market at time $t\in T_0$ with $t\ge\bar{t}$ and $\tilde{z}\in Z^B(f',y')$. Then, we obtain $\pi^{f'}(y'+\tilde{z})>\pi^{f'}(y')$. So, we have $V_{t+3}(f',y'+\tilde{z})>V_{t+3}(f',y')$, which implies that if the firm receives the proposal $\tilde{z}$, then it will accept it. Since $V_{t+3}(f',y'+\tilde{z})\ge w(\pi^{f'}(y'))$, we have $V_{t+3}(f',y'+\tilde{z})>w(\pi^{f'}(y'))$, which implies that if the firm receives the proposal $\tilde{z}$, then it will accept $\tilde{z}$ rather than leave the market. Therefore, by Step 1, for almost every firm that is ready to leave the market at time $t\ge\bar{t}$, if the firm receives the proposal $\tilde{z}$ at time $t+1$, then the firm accepts it.

By Lemma 5, from time $\bar{t}$ on, the consumer can make the proposal $\tilde{z}$ to such firms that are ready to leave the market as many times as he wishes with probability 1, Thus, if his characteristic is $(\bar{u},\bar{x})$ at time t, then he can attain the consumption plan x' with probability 1 under the continuation strategy $\sigma'|_{\bar{h}}$. Therefore,

$$U(\sigma'|_{\bar{h}},\sigma^*|_{-\bar{h}},\bar{h})=\bar{u}(x')>V_t(\bar{c})=U(\sigma^*|_{\bar{h}},\sigma^*|_{-\bar{h}},\bar{h}),$$

which contradicts the definition of equilibrium.

***Case* 2**. $\bar{x}_1<x'_1$.

The proof for this case follows the same way as for Case 1. We omit it.

Q.E.D.

For all $f \in \boldsymbol{F}$, let $\pi^{f*} \equiv \max[f(l)+pl]$. The next proposition is a counterpart of Proposition 2 for firms.

***Proposition* 3.** At time 0, for all $f \in \boldsymbol{F}$, $V_0(f,(0,0)) \geq w(\pi^{f*})$.

***Proof*.** Suppose to the contrary that there exists $\bar{f} \in \boldsymbol{F}$ such that $V_0(\bar{f},(0,0)) \geq w(\pi^{\bar{f}*})$. Since by Assumption 1(P), $d\bar{f}/dl \rightarrow -\infty$ as $l \rightarrow 0^-$, we have $\pi^{\bar{f}*} \underline{>} 0$. Therefore, there exists $y' \in \mathbb{R}_- \times \mathbb{R}_+$ such that $V_0(\bar{f},(0,0)) < w(\pi^{\bar{f}*}(y')) < w(\pi^{\bar{f}*})$ and $(p,1) \cdot y' > 0$.

Step 1. We show that every consumer ready to leave the market at time $t \in T_0$ accepts the trade y'.

For every consumer ready to leave the market at time $t \in T_0$ with the characteristic (u,x), we have $V_t(u,x)=u(x)$. Thus, we have

$$u(x) \geq V_{t+3}(u,x) \tag{1}$$

and by Proposition 2, $x=\arg\max u(x')$ on $B(p,x)$. Since $(p,1) \cdot y' > 0$, the interior of $B(p,(x+y'))$ is non-empty. Thus, by Proposition 2,

$$V_{t+3}(u,x+y') \geq \max u(B(p,(x+y'))). \tag{2}$$

Since $(p,1) \cdot y' > 0$, we have

$$\max u(B(p,(x+y'))) > u(x). \tag{3}$$

Thus, from (2) and (3), we obtain

$$V_{t+3}(u,x+y') > u(x), \tag{4}$$

which implies that if the consumer receives the proposal y', then he accepts it rather than leaves the market. From (1) and (4), we obtain

$$V_{t+3}(u,x+y') > V_{t+3}(u,x),$$

which implies that if the consumer receives the proposal y', then he accepts rather than rejects it.

Step 2. We claim that σ^* is not a subgame perfect equilibrium. Let $\bar{h}=(\bar{f},e_0)$ and we define a strategy profile σ' as follows: for all $s \in T_1 \cup T_2$, for all $h_s=(\bar{h},e_1,e_2,\cdots,e_{s-1}) \in \overrightarrow{H}^B_s$,

$$\sigma_s'(h_s) = \begin{cases} y' & \text{if } s \in T_1 \text{ and } \gamma_s(h_s) = (\bar{f},(0,0)) \\ N & \text{if } s \in T_2 \text{ and } \gamma_s(h_s) = (\bar{f},(0,0)) \\ (0,0) & \text{if } s \in T_1 \text{ and } \gamma_s(h_s) = (\bar{f},y') \\ L & \text{if } s \in T_2 \text{ and } \gamma_s(h_s) = (\bar{f},y'). \end{cases}$$

For any other $h_s = \vec{\bar{H}}_s^B$ with $s \in T_1 \cup T_2$, let $\sigma_s'(h_s) = \sigma_s^*(h_s)$. Then, under the continuation strategy $\sigma'|_{\bar{h}}$, a firm distinguished by the history $\bar{h}$ takes the following actions: (i) before the firm attains the employment-wage plan y', (i-1) if it is chosen as proposer, then it always proposes y', and (i-2) if it is chosen as responder, then it rejects all proposals staying in the market, and (ii) once the firm attains y', (ii-i) if it is chosen as proposer, then it always proposes (0,0), and (ii-2) if it is chosen as responder, then it leaves the market rejecting all proposals.

Since by Lemma 5, the firm can make the proposal y' to a consumer ready to leave the market with probability 1, the firm under $\sigma'|_{\bar{h}}$ attains y' with probability 1. Therefore,

$$W(\sigma'|_{\bar{h}}, \sigma^*|_{-\bar{h}}, \bar{h}) = w(\pi^{\bar{f}}(y')) > V_{\bar{t}}(\bar{f},(0,0)) = W(\sigma^*|_{\bar{h}}, \sigma^*|_{-\bar{h}}, \bar{h}),$$

which contradicts the definition of equilibrium.

Q.E.D.

Now, we are ready to prove the theorem.

Proof of Theorem. Recall that the measure space $(A, \boldsymbol{A}, \nu^A)$ represents the set of consumers and the measure space $(B, \boldsymbol{B}, \nu^B)$ represents the set of firms in the underlying economy. Following Gale (1986a), we introduce a probability measure space $(\Omega, \mathscr{E}, \rho)$ that represents the uncertainty in the matching process.

For all $a \in A$, for all $\omega \in \Omega$, let $\tilde{x}(a, \omega)$ be the terminal consumption plan[14] and $x(a, \omega)$, the actual consumption[15] that

[14]That is, $1 - \tilde{x}_1(a, \omega)$ is equal to the amount of claims for labor that consumer a is liable for and $\tilde{x}_2(a, \omega)$ is equal to the amount of claims for the consumption good held by consumer a.

[15]That is, $1 - x_1(a, \omega)$ is equal to the amount of labor that consumer a actually paid after leaving the market and $x_2(a, \omega)$ is equal to the amount of the consumption good that consumer a actually received after leaving the market.

consumer a would obtain in state ω if he leaves the market at a finite time in ω under σ^*. Then, for every ω and for every consumer a leaving the market at a finite time in state ω, $x_1(a,\omega) \geq \tilde{x}_1(a,\omega)$ and $x_2(a,\omega) \leq \tilde{x}_2(a,\omega)$.[16]

Note that in an equilibrium, for each agent, the probability that a claim that he is concerned with is not fulfilled is zero. Thus, for each a, for $\rho-a.e.$ ω in which consumer a leaves the market, $x(a,\omega)=\tilde{x}(a,\omega)$. For completeness, for any ω and for any consumer a who stays in the market forever in state ω, we put $x(a,\omega)=(0,0)$.

***Claim* 1**. For all $a \in A$, $(p,1)\cdot\int x_2(a,\omega)\,\rho\,(dw) \geq (p,1)\cdot(1,0)$.

Let u^a be consumer a's utility function. Then,

$$V_0(u^a,(1,0))=\int u^a(\tilde{x}(a,\omega)\,\rho\,(dw))=\int u^a(x(a,\omega)\,\rho\,(dw))$$

By strict concavity of u^a (Assumption 1(C)),

$$V_0(u^a,(1,0)) \leq \int u^a(x(a,\omega)\,\rho\,(dw)) \tag{5}$$

where the equality holds only if $x(a,\omega)=\int x(a,\omega)\,\rho\,(dw)$ with probability 1. From Proposition 2 and (5), we prove Claim 1.

For all $b \in B$, for all $\omega \in \Omega$, let $\tilde{y}(b,\omega)$ be the terminal employment-wage plan and $y(b,\omega)$, the actual amounts of employment and wage that firm b would obtain in state ω if the firm leaves the market at a finite time in ω under σ^*. Then, from the same argument as in the case of consumers, for each firm b, for $\rho-a.e.$ ω in which firm b leaves the market, $y(b,\omega)=\tilde{y}(b,\omega)$. For completeness, for any ω and for any firm b that stays in the market forever in state ω, we put $y(b,\omega)=(0,0)$.

***Claim* 2**. For all $b \in B$, if $\varepsilon^B(\mathrm{b})=f$, then $\pi^f(\,y(b,\omega)\,\rho\,d(\omega)) \geq \pi^{f*}$.

[16] Suppose that a firm is liable to pay the consumption good to k consumers when the firm leaves the market but, because some of its claims for labor are not fulfilled, the amount of the consumption good that it can produce is less than the amount that the firm is liable to pay to k consumers. Then, the actual consumptions of k consumers will depend on how to allocate the production of the firm among them. But, since according to our earlier argument, each agent believes that his claims are fulfilled with probability 1, each agent's behavior is not affected by whatever kind of allocation rule is introduced for such a case. So, we assume that for such a case, there is a certain rule known to each agent.

Let f^b be firm b's production function. Then,

$$V_0(f^b,(0,0)) = \int w(\pi^{f^b}(\tilde{y}(b,\omega)))\,\rho(d\omega) = \int w(\pi^{f^b}(y(b,\omega)))\,\rho(d\omega)$$

By strict concavity of w (Assumption 1(E)), we obtain

$$V_0(f^b,(0,0)) \leq w\left(\int \pi^{f^b}(y(b,\omega))\,\rho(d\omega)\right) \tag{6}$$

where the equality holds only if $\pi^{f^b}(y(b,\omega)) = \int \pi^{f^b}(y(b,\omega))\,\rho(d\omega)$ with probability 1. From Proposition 3 and (6), we prove Claim 2.

***Claim* 3**. For $\nu^A-a.e.$ $a \in A$ and for $\nu^B-a.e.$ $b \in B$, Claim 1 and Claim 2 hold with equality, respectively.

By Claims 1 and 2,

$$(p,1)\cdot\int_A\int_\Omega x(a,\omega)\,\rho(d\omega)\,\nu^A(da) + \int_B \pi^{f^b}\left(\int_\Omega y(b,\omega)\,\rho(d\omega)\right)\nu^B(db)$$

$$\geq (p,1)\cdot(1,0)\int_A \nu^A(da) + \int_B \pi^{f^{b*}}\,\nu^B(db) \tag{7}$$

But, for all ω,

$$\int_B y_1(b,\omega)\,\nu^B(db) = \int_A (1 - x_1(a,\omega))\,\nu^A(da),$$

$$\int_A x_2(a,\omega)\,\nu^A(da) = \int_B y_2(b,\omega)\,\nu^B(db).$$

Thus, for all ω,

$$p\int_A x_1(a,\omega)\,\nu^A(da) = p\int_A \nu^A(da) - p\int_B y_1(b,\omega)\,\nu^B(db),$$

$$\int_A x_2(a,\omega)\,\nu^A(da) + \int_B [f^b(y_1(b,\omega)) - y_2(b,\omega)]\,\nu^B(db) = \int_B f^b(y_1(b,\omega))\,\nu^B(db).$$

Therefore, we obtain that for all ω,

$$p\int_A x_1(a,\omega)\,\nu^A(da) + \int_A x_2(a,\omega)\,\nu^A(da) + \int_B [f^b(y_1(b,\omega)) - y_2(b,\omega)]\,\nu^B(db)$$

$$= p\int_A \nu^A(da) + \int_B [f^b(y_1(b,\omega))\,\nu^B(db) - y_1(b,\omega)\,\nu^B(db)$$

$$\leq p\int_A \nu^A(da) + \int_B \pi^{f^{b*}}\,\nu^B(db),$$

where the inequality comes from the definition of $\pi^{f^{b*}}$. Thus, we have

$$(p,1)\int_A x(a,\omega)\,\nu^A(da)+\int_B \pi^{f^b} y(b,\omega)\,\nu^B(db)\leq p\int_A \nu^A(da)+\int_B \pi^{f^{b*}}\nu^B(db). \quad (8)$$

Then, by (8), inequality (7) must hold with equality and by Claims 1 and 2, we obtain

$$\int_A \quad \int_B \quad \int_\Omega$$

$$(p,1)\cdot\int x(a,\omega)\,\rho\,(d\omega)=(p,1)\cdot(1,0) \text{ for } \nu^A-a.e.\ a\in A \quad (9)$$

$$\pi^{f^b}(\int y(b,\omega)\,\rho\,(d\omega)=\pi^{f^{b*}} \qquad \text{for } \nu^B-a.e.\ b\in B \quad (10)$$

***Claim* 4**. For $\nu^A-a.e.$ $a\in A$ and for $\rho-a.e.$ $\omega\in\Omega$, $x(a,\omega)$ maximizes the utility on $B(p,(1,0))$.

It follows from Proposition 2, (5) and (9), that (5) must hold with equality. Thus, for $\nu^A-a.e.$ $a\in A$, $\int x(a,\omega)x(a,\omega)\,\rho\,(d\omega)$ with probability 1. Then, by Proposition 2 and (9), for $\nu^A-a.e.$ $a\in A$ and for $\rho-a.e.$ $\omega\in\Omega$, $x(a,\omega)=\arg\max u(x)$ on $B(p,(1,0))$.

***Claim* 5**. For $\nu^B-a.e.$ $b\in B$, if $\varepsilon^B(b)=f$, then for $\rho-a.e.$ $\omega\in\Omega$, $y_1(b,\omega)=\arg\max\ [f(l)+pl]$ and $y_1(b,\omega)=-py_1(b,\omega)$.

Since f^b is strictly concave (Assumption 1(P)), we have

$$f^{b*} \qquad \pi^{f^b}(\int y(b,\omega)\,\rho\,(d\omega)=\int \pi^{f^b} y(b,\omega)\,\rho\,(d\omega).$$

Thus, from Proposition 3, (6) and (10), we obtain that

$$V_0(f^b,(0,0))=w(\int \pi^{f^b}(y(b,\omega))\,\rho\,(d\omega))=w(\pi^{f^{b*}})$$

That is, (6) holds with equality. Therefore, for $\nu^B-a.e.$ $b\in B$,

$$\pi^{f^b}(y(b,\omega))=\int \pi^{f^b}(y(b,\omega))\,\rho\,(d\omega))=\pi^{f^{b*}} \text{ with probability 1.} \quad (11)$$

By Proposition 1, for $\nu^B-a.e.$ $b\in B$, for $\rho-a.e.$ $\omega\in\Omega$, $(p,1)$ is normal to the supporting line of $Z^B(f,y(b,\omega))$ through 0. Therefore, by (11), for $\nu^B-a.e.$ $b\in B$ and for $\rho-a.e.$ $\omega\in\Omega$, $y_1(b,\omega)=\arg\max$ $[f(l)+pl]$ and $y_2(b,\omega)=-py_1(b,\omega)$.

Q.E.D.

IV. Concluding Comments

We have specifically considered the sequential bargaining in the labor market to examine the relation between the bargaining equilibria and Walrasian equilibria in the case of production economies as well as in the case of the labor market. The theorem serves the dual purpose: sequential bargaining provides a non-cooperative foundation for Walrasian equilibria in the production economies as well as in the labor market.

The model can be extended to the case that each firm employs multiple inputs and produces a single output. But, in the case that a firm produces several outputs, extending the model faces the problem of defining the notion of production surplus. Because there is no obvious way to evaluate different outputs without the price system.

Appendix

***Proof of Lemma* 1.**

Since at each time in T_0, the probability that a firm is matched is less than 1, at each time, there is a positive measure of firms that have never been matched. So, at each time $t \in T_0$, there is a positive measure of firms with characteristics in $\boldsymbol{C}^B_+$.

Suppose to the contrary, that there exists $\bar{t} \in T_0$ such that for all $s \geq \bar{t}$, for $\mu^B_s - a.e.$ $(f,y) \in \boldsymbol{C}^B_+$, $V_s(f,y) > w(\pi^f(y))$.[17] Then, for each firm whose characteristic is in $\boldsymbol{C}^B_+$ at time s, the probability of leaving the market is zero. Thus, for all $(f,y) \in \boldsymbol{C}^B_+$, $V_s(f,y) = -\infty$. But, this contradicts that for any $t \in T_0$, for any $(f,y) \in \boldsymbol{C}^B_+$, $V_t(f,y) \geq w(\pi^f(y)) > -\infty$.

Q.E.D.

***Proof of Lemma* 4.**

***Claim* 1**. There exists $\bar{s} \in T_0$ and $\bar{\mathbf{C}} \subset \boldsymbol{C}^B_+$ such that $\mu^B_{\bar{s}}(\bar{\mathbf{C}}) > 0$ and for all $(f,y) \in \bar{\mathbf{C}}$, $y_1 < 0$ and $V_{\bar{s}}(f,y) > w(\pi^f(y))$.

Suppose to the contrary, that for all $t \in T_0$, for $\mu^B_t - a.e.$ $(f,y) \in \boldsymbol{C}^B_+$ with $y_1 < 0$, $V_t(f,y) > w(\pi^f(y))$. Then, for almost every firm, the

[17] It is clear that for all $t \in T_0$, $V_t(f,y) \geq w(\pi^f(y))$ if $(f,y) \in \boldsymbol{C}^B_+$. For, a firm can leave the market with the current employment-wage plan y with probability 1.

probability that it leaves the market with some claims for labor is 0. Thus, almost every firm employs no labor, and almost every consumer leaves the market with the terminal consumption plan (1,0).

Pick a firm ready to leave the market at time $t' \in T_0$ with characteristic $(f,(0,y_2))$ where $y_2 < 0$. Then, for each consumer ready to leave the market at time $t'' \geq t'$ with characteristic $(u,(1,0))$, there exists $z \in Z^B(f,(0,y_2))$ such that $u((1,0)+z) > u(1,0)$. Thus, $V_{t''+3}(u(1,0)+z) > u(1,0) = V_{t''}(u(1,0)) \geq V_{t''+3}(u(1,0)$, which contradicts Lemma 3(i).

***Claim* 2**. For all $t \in T_0$, for $\mu^A_t - a.e.$ $(u,x) \in \mathbf{C}^A_+$ with $x \in \partial(\mathbb{R}^2_+)$, $V_t(u,x) > u(x)$.[18]

By Assumption 5, no disqualified consumer is matched, and therefore can not leave the market. Thus, for all $t \in T_0$, for all u, $V_t(u(0,0)) = -\infty$. Hence, we need consider only characteristics whose current consumption plans are not (0,0). Recall that $\bar{s}$ and $\bar{\mathbf{C}}$ are the time and the set of characteristics referred to in Claim 1, respectively.

***Case* 1.** $t \leq \bar{s}$.
Suppose to the contrary, that there exist $t' \leq \bar{s}$ and $\mathbf{C}' \subset \mathbf{C}^A_+$ such that $\mu^A_{t'}(\mathbf{C}') > 0$ and for all $(u,x) \in \mathbf{C}'$, $x \in \partial(\mathbb{R}^2_+) \setminus \{(0,0)\}$ and $V_{t'}(u,x) \in u(x)$. Then, since $u(x) = 0$ for all $(u,x) \in \mathbf{C}'$ by Assumption 1(C), $Z^A(u,x) \cap Z^B(f,y) \neq \varnothing$ for all $(u,x) \in \mathbf{C}'$, for all $(f,y) \in \bar{\mathbf{C}}$. Thus, there exists $z \in Z^A(u,x)$ such that $\pi^f(y+z) > \pi^f(y)$. So, we obtain $V_{\bar{s}+3}(f,y+z) > V_{\bar{s}+3}(f,y)$, which contradicts Lemma 3(ii).

***Case* 2.** $t > \bar{s}$.
Suppose to the contrary, that there exist $t' > \bar{s}$ and $\mathbf{C}' \subset \mathbf{C}^A$ such that $\mu^A_{t'}(\mathbf{C}') > 0$ and for all $(u,x) \in \mathbf{C}'$, $x \in \partial(\mathbb{R}^2_+) \setminus \{(0,0)\}$ and $V_{t'}(u,x) \in u(x)$. Then, since for all $(u,x) \in \mathbf{C}'$, for all $(f,y) \in \bar{\mathbf{C}}$, there exists $z \in Z^B(f,y)$ such that $\pi^f(y+z) > \pi^f(y)$, we obtain $V_{t'+3}(u,x+z) > V_{t'+3}(u,x)$, which contradicts Lemma 3(i).

Suppose that Lemma 4 is not true, that is, for all $t \in T_0$, for $\mu^A_t - a.e.$ $(u,x) \in \mathbf{C}^A_+$ with x lying in the interior of $[0,1] \times \mathbb{R}_+$, $V_t(u,x) > u(x)$. Then, by Claim 2, almost every consumer is ready to leave the market with a characteristic $(u,(1,x_2))$ where $x_2 \geq 0$. Thus, the mean supply of labor in the economy is equal to zero, which contradicts Claim 1 that implies that the mean employment level should be positive.

[18]For any subset E of $\mathbb{R}^2$, ∂E denotes the set of boundary points of E.

Q.E.D.

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