Sustainable Emission Control Policies: Viability Theory Approach

Kunhong Kim and Jacek B. Krawczyk

Our interest is in the relationship between the environment and economic growth. Because various interest groups see this issue differently, the typical optimization approach based on representative agent is not suitable. This is mainly because assessing the relative weight between consumption and environment in the utility function in a democracy is a sensitive political process. On the other hand, *constraints* on capital, consumption, and pollution levels should be agreed considerably easier than the aforementioned weight because the constraints refer to quantifiable measures. We propose that a regulator can look for a feasible strategy for emission control that will maintain capital, consumption, and pollution in a closed set of constraints. Such a strategy is called *viable* in viability theory. Viability theory is the study of dynamic systems that asks what set of initial conditions will generate evolutions that obey the laws of motion of a system and remain in a certain state constraints set for the duration of the evolution. We apply viability theory to a neoclassical model to identify which current economic states are sustainable under smooth adjustments of abatement-rate in the future. Among many observations, we note that countries that embark on an ambitious abatement program may fail to maintain their economies within the state constraints if their present levels of capital and consumption are low.

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I. Introduction

The atmospheric concentration of ultrafine dust is a major socio-environmental issue in South Korea. Government policymakers are increasingly facing conflicting challenges on emission abatement and the maintenance of economic livelihood of residents. Our aim is to analyze the basic problems that economies face in setting the macro-policies of abatement. We use viability theory of Aubin *et al.* (2011) to study the attainability of *sets* of desired economic states. We believe that viability theory serves our aim better than optimization that is routinely concerned with strategies that steer an economy toward a *single* state.

Viability theory is the study of dynamic systems that asks what set of possible paths obeys the system's laws of motion and remains in a certain state-constraint set. Our use of viability theory in this study will enable us to answer the following question: *What are the current consumption, capital, and emission-control levels that can be sustained if all we know are equations of motions for these economic variables and that the emission-control adjustments will be limited?*[1] An optimal policy may demand arbitrarily large control adjustments that may be financially unrealistic for a poor country. Therefore, *smooth* adjustments, which can be predetermined, are a realistic assumption about the applicability of a strategy that is calculated as a *viable* solution. The usual perfect foresight analysis would specify one future saddle-point path for consumption and capital. The viability theory approach relaxes this assumption and proposes that many paths can be followed through appropriate abatement policies as long as such paths are near the saddle point.

For example, suppose that we know the current emission-control level. Under the classical approach, only one consumption and capital combination would likely be *viable* (*i.e.*, one equilibrium path could originate from this combination). In this case, viability reduces to equilibrium. By contrast, a viability analysis can establish the set of all pairs of consumption and capital (k, c) such that the evolutions

[1] We note that a similar question referring to taxation policy instead of emission control was answered through viability theory in Krawczyk, and Judd (2015).

which originate from each pair remain in a predetermined state-space constraint set, for certain future emission-control paths. We assert that the collection of all such initial conditions, which we call the *viability kernel*, generalizes the notion of equilibrium, whereby this generalization is one of the themes of viability theory.

Overall, provided that the model calibration is believable, we contend that studying the implications of various emission-control regimes in this setting is a beneficial exercise for politicians and economists.

Viability theory is part of a set-valued analysis (see Cardaliaguet *et al.* 1999). This theory's main machinery consists of the formulation and solution of dynamic problems in terms of differential *inclusions* rather than equations. This provides an additional piece of motivation for using viability theory in economics. Economic agents *cannot* perfectly solve the Euler equation at all times. We believe that, in reality, movements are constrained by optimality conditions but cannot be pinned down precisely because of parameter uncertainty and numerical errors, among others. Hence, differential inclusions, which enable multiple economic paths, are a superior method to represent economic dynamics. Viability theory, which defines dynamic systems as differential inclusions, is the appropriate mathematical tool to deal with the "imprecise" optimality conditions. These conditions may occur in many economic problems. The presentation of viability theory in the context of the neoclassical growth-abatement model, which is a relatively well-known model, should encourage economists to use viability theory and apply it to other macroeconomic problems.

The process of solving viability problems is computationally intensive. However, thanks to some specialized software, solving simple models, at present up to five state variables and two controls, is possible. The software we use is VIKAASA (see Krawczyk, and Pharo 2014a, 2014b).

The rest of this paper is organized as follows. Section Ⅱ provides a brief and intuitive introduction to viability theory. Section Ⅲ presents a prototype neoclassical model with environmental control and compares the optimization-based approach with the viability theory-based approach. Section Ⅳ discusses the calibration of the parameter values, explains how we calculated the viability kernel, and discusses the results of the viability analysis, thereby enabling us to propose a policy advice. Section Ⅴ provides the concluding remarks.

II. Short Introduction to Viability Theory

Viability theory is used extensively in this study. This theory is a relatively new area of continuous mathematics that has already enjoyed considerable success among ecologists (see Martin 2004; Doyen, and Pereau 2012; Aubin *et al.* 2011, Chapter 7). The main idea behind viability theory is to provide a framework for the computation of strategies to achieve *desirable* outcomes rather than model-optimal ones.

Viability theory provides a method of accounting for the complexities of dynamic systems when determining whether a given course of action is sustainable or not. In particular, viability theory proposes *satisficing* solutions (in the sense of Simon 1955) to prevent those systems from becoming unsustainable. Below, we provide a short introduction to viability theory and highlight the novelty of this approach to decision-making.

Viability theory has significant potential in macroeconomics, as shown in the initial work by the authors of the current study (and others) in Krawczyk, and Kim (2009); Krawczyk, and Serea (2013); Krawczyk, and Kim (2014); Bonneuil, and Boucekkine (2014); Filar *et al.* (2015). Krawczyk, and Judd (2015) used viability theory to propose tolerable levels of taxation rates, which would provide a certain level of government service without stifling economic activity or bankrupting government.

The current study models and solves another socioeconomic problem that is principally concerned with sustainability in a holistic sense. Specifically, we deal with strategies that assure socially acceptable trade-offs between environmental quality and economic growth.

Rigorous introductions to viability theory are discussed in Aubin (1991); Veliov (1993); Quincampoix, and Veliov (1998); Aubin *et al.* (2011). Here we present only those notions of viability theory that are essential to our analysis.

Consider a dynamic system with n state and m control variables. Let $x(t)$ be a vector of values of n state variables and let $u(t)$ be a vector of values of m control variables at time t. In viability theory, the basic description of a dynamic system is the following differential inclusion:

$$\dot{x}(t) \in F(x(t)) . \tag{1}$$

At state $x(t)$, the change in the system's state (*i.e.*, its velocity) will be a member of $F(x(t))$, where F is a set-valued map from the system states to the sets of possible velocities. In control theory, map F has the form $F(x) = f(x, U) = \{f(x, u);\ \ u \in U\}$, where $f : \mathrm{R}^n \times U \mapsto \mathrm{R}^n$ is a continuous vector-valued function that represents the system's equations of motion and U is a compact set in R^m. In this case, we can re-write Formulation (1) as follows:

$$\dot{x}(t) = F(x(t),\ u(t)) \tag{2}$$

$$u(t) \in U(x(t)), \tag{3}$$

where Equation (2) is a standard parameterized differential (vector) equation and Inclusion (3) states that the control choice $u(\cdot)$ must come from a set $U(x(\cdot))$, which may be state-dependent.

Let K represent the closed set of constraints that state $x(t)$ must satisfy for all t. Given a set-valued map $F : K \to \mathrm{R}^n$, we say that $x_0 \in K$ is *viable in K under F* if at least one solution to the following system exists:

$$\forall t \in \Theta \begin{cases} x(t) \in K \\ \dot{x}(t) \in F(x(t)) \end{cases}, \tag{4}$$

which starts at $x(0)=x_0$ and remains in K forever: $\Theta \equiv [0, \infty)$.

Formulation (4) describes the viability of an individual system state. The viability kernel $\mathcal{V}_F(K)$ is the set of all viable states:

$$\mathcal{V}_F(K) \equiv \\ \{x(0) : \exists\ x(t) \text{ satisfying Formulations (2)–(3) and constraint } K\ \forall t\}. \tag{5}$$

For a control problem, the viability kernel $\mathcal{V}_F(K)$ is the area of K in which a control exists that can indefinitely maintain the system within K. If a trajectory begins inside the viability kernel $\mathcal{V}_F(K)$, then we have sufficient controls to maintain this trajectory in the constraint set K for all t. If a trajectory begins outside the kernel, then it will inevitably leave K. The viability kernel $\mathcal{V}_F(K)$ has important implications for policy: it allows us to formulate strategies that maintain the system's viability.

In this study, the viability kernels and "satisficing" strategies are computed for the socio-environmental dynamic problem in economics. Knowing the kernel, we will then be able to advise on sustainability and environmental conservation.

III. Neoclassical Model of Production and Emission

We sketch the typical optimization-based approach and discuss the merits of the alternative approach based on viability theory. We use Stokey (1998) as our benchmark optimization model.

A. Optimization-Based Approach

Typical optimization-based approach assumes an economy populated by infinitely lived, representative agents. For simplicity of presentation, population growth is taken to be zero. The preference of the representative agent is given by the following utility function:

$$\int_0^\infty e^{-\rho t}\left[\frac{c^{1-\sigma}-1}{1-\sigma}-\frac{B}{\gamma}X^{\gamma}\right]dt. \quad (6)$$

The representative agent's utility depends positively on consumption c and negatively on aggregate pollution level X. The parameters are ρ, σ, γ, and B, where ρ is the rate of time preference, σ is the relative risk aversion parameter, and B determines the relative weight of c and X in the agent's utility.

Output is produced using a neoclassical production function. In per capita terms, the production function is $y = A^{1-\alpha}K^{\alpha}$, where y is the per capita output, k is the per capita capital stock, and A is the exogenously given labor-augmenting technology, which is constant in our model.[2]

A negative externality is present. That is, when they make a market decision, private agents treat total pollution level X as a given, thereby causing too much pollution. In this situation, a social planner (*i.e.*, government) imposes emission control. Emission control is captured by variable z = 1 – θ, where θ represents the proportion of output y devoted to abatement. $z \in [0, 1]$. More abatement (*i.e.*, higher θ; therefore, lower

[2] Similarly to Lee (2012), we assume that the level of technology is given and fixed. While Lee's study was mainly based on comparative steady states, ours concerns transitional dynamics toward a steady state.

z) reduces pollution X. This relationship is captured by $X = A^{1-\alpha}K^{\alpha} \cdot z^{\beta}$, where $\beta > 1$. Abatement has a positive but diminishing marginal impact on pollution reduction because $\beta > 1$. However, pollution reduction effort takes away resources for abatement. The output net of abatement cost, which can be used for consumption or capital accumulation, is then given by:

$$y = A^{1-\alpha}k^{\alpha} \cdot (1-\theta) = A^{1-\alpha}k^{\alpha}z. \tag{7}$$

A tradeoff occurs between consumption and environment because of this emission control technology. Social planner selects a series of z to maximize the representative agent's infinite horizon utility.

Given this emission control technology z, the social planner maximizes the representative agent's infinite horizon utility:

$$\int_0^{\infty} e^{-\rho t}\left[\frac{c^{1-\sigma}-1}{1-\sigma} - \frac{B}{\gamma}(A^{1-\alpha}k^{\alpha}z^{\beta})^{\gamma}\right]dt,$$

subject to the economy-wide resource constraint:

$$\dot{k} = A^{1-\alpha}k^{\alpha}z - \delta k - c.$$

Since the emission-control technology is bound between zero and one, $z \in [0, 1]$, emission control is not binding (*i.e.*, $z = 1$) in some region. When the existing capital stock is low and the shadow price of capital is high, it is optimal not to impose any emission control. This case makes the 1st order condition for social planner optimization quite complex. Details of the necessary conditions of the social planner optimization and the dynamics generated from such optimization appear in Stokey (1998).

B. Viability Theory-Based Approach

To apply the social planner optimization, it is crucial that the planner knows the relative weight between consumption and environment in the representative agent's utility function (This value is captured by parameter B in the utility function shown in Equation (6)). The process of assessing a particular value of disutility from the pollution level X in a democracy is an extremely sensitive political process because different interest groups view this disutility differently. By contrast, the constraints on capital, consumption, and pollution levels should be agreed much easier because they are expressed in quantifiable units. The regulator can look for a *viable* strategy $z \in [0, 1]$, which will retain k, c, and X in a closed set of constraints K. This result would be a *satisficing* solution in that the emissions would be capped without hurting consumption too much.

We contend that implementing a viable solution will be a more socially straightforward process than implementing an optimal solution. Moreover, parameterizing the former in the upper/lower bounds of K will enable the regulator to investigate various trade-offs between those bounds and the level of emission controls.

We apply the viability approach to the problem of environment and growth in the following manner.

Given aggregate pollution level X and emission control z imposed by the government, the private agent maximizes the following infinite horizon utility:

$$\int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} dt,$$

subject to the resource constraint $\dot{k} = A^{1-\alpha} k^{\alpha} z - \delta k - c$. Of course, private agents care about the aggregate pollution level but each individual's impact is miniscule. Hence, they cannot do anything about pollution on their own. Consequently, they treat the status of the environment as a given. Therefore, environmental concerns do not play any role in their individual decisions concerning consumption–saving. The first-order conditions of the private agent's optimization are as follows:

$$\dot{c} = \frac{c}{\sigma}(\alpha A^{1-\alpha} k^{\alpha-1} z - \delta - \rho) \tag{8}$$

$$\dot{k} = A^{1-\alpha} k^{\alpha} z - \delta k - c. \tag{9}$$

Equation (8) shows the Euler equation for the private agent's consumption choice over time (*i.e.*, optimal savings decision). Consumption increases as long as the real interest rate ($\alpha A^{1-\alpha} K^{\alpha-1} z - \delta$) exceeds the time preference rate ρ. Evidently, consumption may slide toward the trivial steady state of zero if depreciation or the consumer's time preferences are large relative to the marginal productivity of capital. Equation (9) is the resource constraint. Net investment is positive (*i.e.*, capital stock increases) when the output net of abatement cost $A^{1-\alpha}K^{\alpha}z$ is larger than the sum of the capital stock depreciated δk and consumption level *c*.

The social planner controls *z* to maintain *k*, *c*, and *X* within bounds considering the dynamics implied by the private agent's intertemporal choice (see Equations (8) and (9)). One of the social planner's concerns is to limit the level of pollution (*e.g.*, due to health hazards):

$$X = A^{1-\alpha}k^{\alpha}z^{\beta} \leq \bar{X}, \tag{10}$$

where $\bar{X} > 0$ is the pollution limit.

The system's dynamics is completed by a differential inclusion for emission control:

$$\dot{z} = u \in [-d, d] \equiv U, \qquad d \geq 0, \tag{11}$$

where *u* is the speed of the emission-control adjustments and can be selected as slow when $|d|$ is small and fast if $|d|$ is large. Presumably, a poor country will select a small $d < 0$ to only gradually strengthen emission control.

The four relationships (Equations (8), (9), (10), and Inclusion (11)) jointly constitute the basic representation of the economy at hand, which is referred to as map *F*.

We recognize that this system is nonlinear and with multiple steady states. We can also see that the dynamics (*i.e.*, Equations (8), (9), (10), and Inclusion (11)) will be difficult to control through *z* – actually *u* – because of the role of *z*. Equation (9) indicates that increasing *z*, that is, decreasing abatement, will accelerate capital growth. However, Equation (8) shows that this will not necessarily increase consumption because the marginal product of capital is low for large capitals. Consequently, the parenthesized term in Equation (8) may be negative and consumption will decrease.

This system's dynamics is even more complex due to the emission constraint (*i.e.*, Equation (10)). Decreasing *z* diminishes emission but it also slows down capital growth and, unless capital is very low, has a negative impact on consumption growth.

Economic and social theories can shed light on which economic states (k, c, z) are admissible. For example, we know that if capital is above the *golden rule* steady state, that capital level is inefficient. Evidently, capital must be positive for subsistence level of consumption. Consumption must be positive for subsistence and cannot be requested to be above the level which corresponds to the capital golden rule steady state.

The preceding information can define admissible (or efficient) *normative* constraints for the economy at hand. The *modal* constraint $z \in [0, 1]$ completes the constraint set K, within which the planner may request the economy to remain. Such K is as follows:

$$K \equiv \{(k, c, z) : k \in [\underline{k}, \bar{k}], \ c \in [\underline{c}, \bar{c}], \ z \in [0, 1], \ A^{1-\alpha} k^{\alpha} z^{\beta} \le \bar{X}\}. \quad (12)$$

Given the constraint set K and the system's dynamics F, we want to establish the viability kernel $\mathcal{V}_F(K)$ (*i.e.*, the loci of economic states from which control adjustments from U can guarantee that the economy remains in K).

IV. Viability Simulations

A. Calibration of Parameters

We assume that $\rho = 0.04$, $\alpha = 0.4$, $\sigma = 2$, and $\delta = 0.1$ broadly characterize a reasonably industrialized economy composed of rational agents interested in the near future. Notably, for $\rho = 0.04$, $\exp(-0.04 \cdot 10) = 0.67$, and $\exp(-0.04 \cdot 50) = 0.13$ so, what can happen in 50 years is approximately 5 times less important than what will happen in 10 years.

We normalize the unit of measurement such that the steady state value of capital stock k_{ss} is equal to 1 for the no-abatement case $z = 1$. This enables us to calibrate the productivity level A. Setting $\dot{c} = 0$ with $k_{ss} = 1$ and $z = 1$ in Equation (8) yields the following:

$$A^{1-\alpha} = \frac{\delta + \rho}{\alpha} = 0.35.$$

Additionally, with $k_{ss} = 1$ and $z = 1$, Equation (9) implies that the steady state value of consumption c_{ss} is as follows:

$$c_{ss} = A^{1-\alpha} - \delta = 0.25.$$

Furthermore, the steady state values of output y_{ss} and investment i_{ss} are $y_{ss} = A^{1-\alpha}k_{ss}^{\alpha} = A^{1-\alpha}$ and $i_{ss} = \delta k_{ss} = \delta$, respectively, with $z = 1$. Therefore, with the calibrated parameter values with $z = 1$, $c_{ss}/y_{ss} = 0.71$, $i_{ss}/y_{ss} = 0.29$, and $k_{ss}/y_{ss} = 3$. These ratios match quite well with the stylized facts for industrial countries, thereby providing us with some confidence in our calibrated model.

We assume that $\beta = 3$.[3] The no-abatement steady-state emission is $X = A^{1-\alpha} = 0.35$.

Now, we need to set the normative boundaries to define K introduced in Section III. In a "real world" calibration, constraints come from a combination of normative and modal sources, as well as from the computational requirement to close K. For example, the lower bound on capital may be tied to a normative requirement concerning a country's GDP, whereas the upper bound may be based on the size of the capital stock that should not exceed the golden rule steady state level. Bounds on consumption can be similarly determined. In general, normative requirements may be determined through certain auxiliary optimization procedure or may be externally given (*e.g.*, politically). We present the following proposals:

- **capital** should be between:
  - 10% of the steady state capital stock without emission control k_{ss} and
  - the value slightly above the golden rule level of the capital stock[4]
  - that is, $k \in [0.1, 1.25]$;
- **consumption** should range between:
  - 1/25 of the steady-state consumption without emission control c_{ss} and
  - the value slightly above the golden rule level of consumption

[3] The value used in Section IV of Stokey (1998).

[4] Equation (9) with $\dot{k} = 0$ and $z = 1$ implies that $c = A^{1-\alpha}k^{\alpha} - \delta k$. c is maximized when

$$k = \left(\frac{\alpha A^{1-\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}}$$

which is the golden rule level of capital stock. Given the calibrated parameter values, it is equal to 1.2237. The corresponding golden rule consumption level is 0.2571.

- that is, $c \in [0.01, 0.26]$;
- **emission level upper bound** is set at approximately 80% of the emission level at the steady state without emission control, that is, we select 0.275, which is approximately 80% of 0.35.
- **emission-control adjustment speed** *i.e.*, the amount by which the regulator can change the current control level within a year will be between −25 and 25 percentage points; hence, $u \in [-0.25, 0.25]$, where u is the emission-control adjustment speed.

As said in Section III, the modal constraint for z is $z \in [0, 1]$. Therefore, the constraint set K, for which we will seek to compute the viability kernel, is as follows:

$$K \equiv \{(k, c, z): k \in [0.1, 1.25], c \in [0.01, 0.26], \\ z \in [0, 1], k^{0.4}z^3 \leq 0.7857\}.$$[5]

The viability problem is then to determine the kernel $\mathcal{V}_F(K) \subset K \subset \mathrm{R}^3$ for the dynamics F, which is defined through the vector differential inclusion (*i.e.*, Equations (8), (9) and Inclusion (11) combined). We will use VIKAASA to compute $\mathcal{V}_F(K)$.

To facilitate our interpretation of the viability simulation results presented in the later section, we also consider another constraint set as follows:

$$K_u \equiv \{(k, c, z): k \in [0.1, 1.25], c \in [0.01, 0.26], z \in [0, 1]\}.$$

This is the constraint set without emission upper bound imposed. In Section *C*, we will compute the corresponding viability kernel $\mathcal{V}_F(K_u) \subset K_u \subset \mathrm{R}^3$ and the kernel $\mathcal{V}_F(K)$. Given that $K \subset K_u$ and the system's dynamics are the same F for each problem, the constrained emission viability kernel will be a subset of the unconstrained emission viability kernel $\mathcal{V}_F(K) \subset \mathcal{V}_F(K_u)$.

B. Method for Determining Viability Kernels

We used VIKAASA[6] to compute the viability kernel approximation

[5] 0.7857 comes from 0.275 divided by $A^{1-\alpha} = 0.35$.

[6] See Krawczyk, and Pharo (2014a, 2014b).

for the problem considered. We check the viability of each $x^h \in K^h \subset K$, where K^h is a discretized K. For each $x^h \in K^h$, VIKAASA assesses whether a dynamic evolution originating at x^h can be controlled to a nearly steady state without leaving the constraint set in finite time. The points that can be brought sufficiently near such a state are included in the kernel, whereas those that are not are excluded.

Points that violate the transversality conditions are excluded from the kernel because only points that can be controlled to a nearly steady state without leaving the constraint set are included in the viability kernel.

C. Simulation Results

a) (k, c)-Dynamics

Figure 1 illustrates the calibrated system's movements, which presents the vector fields ("quiver") in the *capital–consumption* state space for two different emission control levels, namely, $z = 1$ and $z = 0.5$.

The steady states for a given value of z are shown as the large green dot in each panel. In each panel, the locus of the points along which $\dot{k} = 0$ (resp. $\dot{c} = 0$) is shown by the curved (resp. vertical) red dashed line. The saddle-path is the black dashed line that goes through the steady state point and has a slope between those of the $\dot{k} = 0$ and $\dot{c} = 0$ lines at the steady state. In each panel, we observe that the closer we are to a steady state, the slower the system will be moving. It therefore appears that there should exist choices of z that are capable of changing the system's directions from *away from the saddle-path* to *toward the saddle-path*.

We can also expect that certain k, c combinations will not reach the saddle-path. For example, for $z = 0.5$, large consumption and low capital may generate evolutions that will move to the left and cross the lower bound of the capital.

The optimization approach treats k as the state variable and c as the jump variable. Current and all future values of the control variable z are considered to be known to economic agents. For the given values of the state variable k, the initial value of the jump variable c is selected, such that the economy immediately moves to a point on the saddle-path. Thereafter, the economy moves along the saddle-path and converges to the steady state.

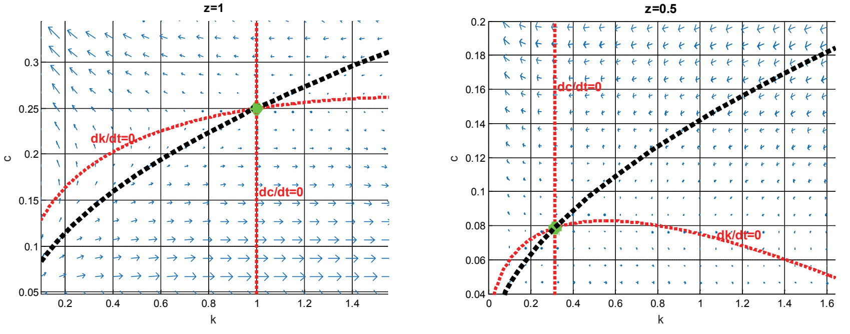

Figure 1
k, c Vector Fields for $z = 1$ and $z = 0.5$

b) Viable and Non-Viable Evolutions

In the viability theory-based approach, k, c, and z are state variables. Economic agents only know the current value z that is controlled by u. The policymaker (social planner) treats the first order conditions of private economic agents as a given and selects u to smoothly adjust z over time to retain k, c, and z in the state constraint set. However, a smooth adjustment of z will exist only for (k, c) that belongs to the viability kernel.

It will be evident from analyzing the viability kernels, calculated below, from which states one can generate evolutions terminating at a steady state and from which one cannot. The states from which the economy can be stabilized belong to the viability kernel and are called viable. We classify the other states as non-viable.[7]

The application of VIKAASA enabled us to obtain kernels $\mathcal{V}_F(K)$ and $\mathcal{V}_F(K_u)$, see Figure 2 where "boulders" represent the viability kernels.

A few explanatory remarks on the method of interpreting the boulders may be necessary. The dimensionality of our problem is 3 (*i.e.*, k, c, and z). Emission X is a also a variable of interest in the problem (and, to a limited extent, output). These values (*i.e.*, k, c, z, X, and output) are computed using VIKAASA and kept in a 5D array. Hence, the 3D "boulders" are convex hulls spanned in 3D (and later, 2D) slices

[7] Our computational method will miss some viable points if they are viable only because the evolutions starting at them are large orbits (or cycles). However, we did not encounter similar points in our experiments.

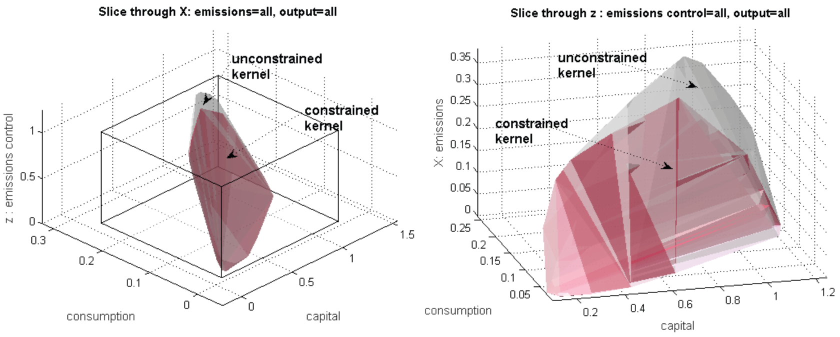

FIGURE 2
VIABILITY KERNEL

through the 5D array. In the picture titles in Figures 2, 3, and 9, we list the variables through which the 5D array is sliced.

The left panel in Figure 2 displays the kernel in k, c, and z for all admissible values of emission and output. The gray (noticeably larger) boulder in this panel represents the viability kernel $\mathcal{V}_F(K_u)$, which corresponds to the set of constraints K_u. This kernel is encased by the constraint "box" K_u. The purple (noticeably smaller) boulder is contained in the gray boulder and represents the constrained emission kernel.

In the right panel, the same color convention is applied. The boulder is drawn in k, c, and X for all admissible values of the emission control and output. Emission X is on the vertical axis for which we do not draw the upper limit because X is unbounded for the unconstrained emission problem.

For completeness of the analysis, we have also computed the emission values X for when there is no emission constraint, see the gray part of the slice in the right panel. This slice will be useful in Section d), where we will discuss the possibility of the transition of an emission-unconstrained economy to an emission-constrained economy.

We see that the purple slice in each panel is a subset of the corresponding gray slice as $\mathcal{V}_F(K) \subset \mathcal{V}_F(K_u)$.

We first observe that the kernel in the left panel is "thin" and occupies a relatively small space within the constraint set K_u, which is represented by the "box". This kernel's size suggests that there are numerous combinations of (k, c, z) for which there is no path of z resulting from admissible adjustments $u \in U$, that could keep the economy in K.

We also notice that the kernel becomes moderately "thicker" for low z (*i.e.*, for stricter emission control). This tells us that there are more viable k, c combinations for economies that engage in strong emission control than for those that do not.

The right panel confirms that unabated emission will not exceed 0.35 and many economic states have emission levels below 0.275 *i.e.*, the imposed upper limit on emission.

In general, we observe that viable consumption must be rather closely aligned with capital. This observation should be expected because plausible consumption strongly depends on capital.[8]

Of particular interest are some qualitative features of evolutions that originate on each side of the kernel's frontier. Consider the (k, c) and (k, z) slices of the viability kernel shown in Figure 3. The left panel is a slice-through of all the z, X, and output values while the right panel is a slice-through of all the c, X, and output values.

In the left panel, we again observe that (for viability) consumption has to be somewhat proportional to capital. In contrast, in the right panel, nearly all states, except those characterized by large capital and lax emission controls, can be viable. For viability, a point must evidently lie in both slices.

Let us look at the viability of a state in which capital is small, consumption is medium-small, and emission control is weak. Consider the state (k, c, z) = [0.2769, 0.1062, 0.8846] — Case (1) (see the parenthesized numbers in Figure 3). This point may be typical of a less industrialized (but not starving) country, which starts strengthening emission control. The corresponding evolution from this point is represented by the dashed lines in Figure 3 and in the time profiles in Figure 4.

The time profile in Figures 4–8 display the time evolutions not only of k, c, z, and X, but also of output,[9] emission control adjustments u, and "velocity". The velocity is the Euclidean norm $\sqrt{\dot{k}^2 + \dot{c}^2 + \dot{z}^2}$, which informs us about the steadiness of the economy. Evidently, the

[8] We observe that they form a narrow viability "corridor" and unlike the relationship between z and k in the right panel of Figure 3. We attribute the former to the necessary proximity of (k, c) to the saddle-path, already manifested in the thinness of the viability kernel in Figure 2, left panel.

[9] Output (net of abatement cost) is a composite of k and z and behaves similarly to emission X.

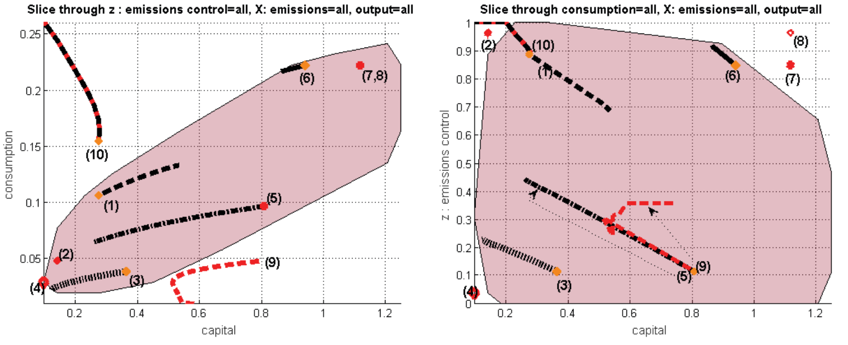

Figure 3
Evolutions in the 2D Viability Kernel's Slices

closer the velocity is to zero, the closer the economy is to a steady state. Unsurprisingly, all velocities converge to zero in Figure 4 where viable evolutions are plotted, whereas they fail to converge to zero in Figure 5 where non-viable evolutions are plotted.

Increasing emission control from [0.2769, 0.1062, 0.8846] is very successful: capital and consumption grows and emission diminishes. Such is the case because decreasing z in this economy characterized by low k and c can still generate a growth of capital and consumption.

Consider now a state with even less capital and less abatement (*i.e.*, [0.1442, 0.0485, 0.9615] — Case (2)). The consumption level is also low. This state is shown as a red dot on the left part of each slice in Figure 3. The dot is inside the 2D (k, c) slice (Figure 3, left panel bottom part). Nevertheless, this state is shown as nonviable (see the 2D (k, z) slice in Figure 3, right panel upper part). We want to explain why this is so.

In Figure 5, emission control causes consumption to diminish (in period 53) to below the lower limit as depicted by the red dashed–dotted line. Prior to this period, the economy has not stabilized. This is a "sad" scenario of a poor country that wants to grow and control emission. Capital productivity is high for low capital and capital grows rapidly. However, increasing emission control (low z) and capital diminishes the parenthesized term in Equation (8), which becomes smaller than depreciation augmented by time preference ($\delta + \rho$).[10]

[10] Keeping $z = 1$ improves consumption but capital grows above the upper bound.

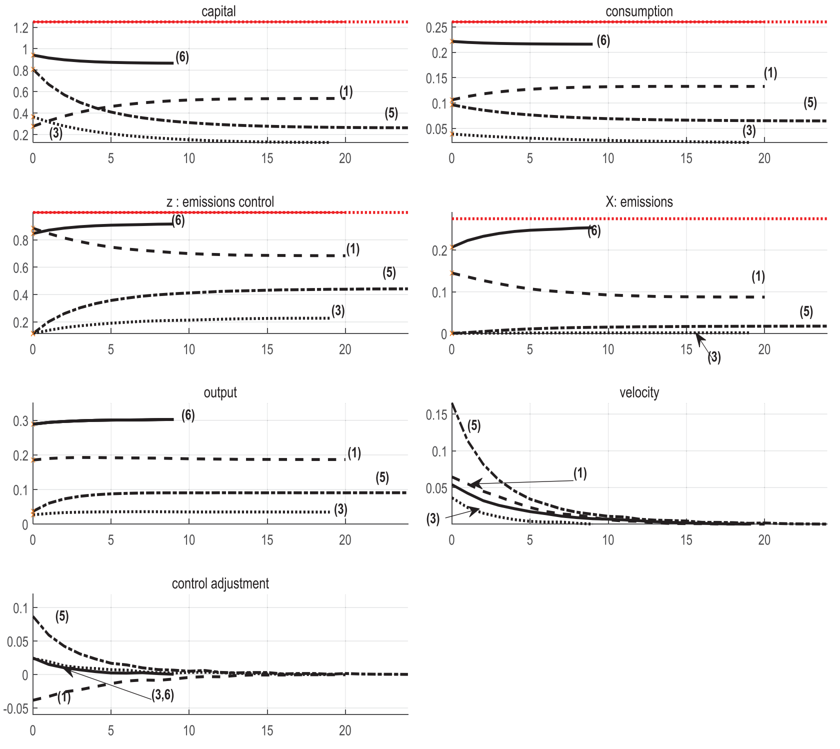

Figure 4
Time Profiles of Viable Evolutions

We now discuss viable evolution of an economy with low capital and consumption but with ambitious emission control. Let us characterize this economy by the state [0.3654, 0.03885, 0.1154] — Case (3), inside each slice in Figure 3, close to the left bottom corner. This state is viable (see the dotted lines in Figures 3 and 4). The emission control program can be maintained but at the cost of decreasing the already low capital and consumption.

An even poorer and more ambitious economy can be characterized by the state [0.1, 0.02923, 0.0385] — Case (4). This state is represented by the diamond in Figure 3. This state is clearly nonviable. Figure 5 illustrates that the emission control program drains capital in one period as depicted by the red dotted line.

Even well-capitalized economies (*e.g.*, [0.8077, 0.0956, 0.1154] — Case (5)), may end up poorer if an ambitious abatement program is implemented. Figures 3 and 4 mark the corresponding evolution using a black dashed–dotted line. Emission control is relaxed (z increases) because emission is minimal with the original small z. This scenario does not stop capital from decreasing because z grows gradually and consumption is still relatively high. Consumption also decreases but the drop is small because capital productivity matches $\delta + \rho$ fast. As a result, this economy stabilizes with low emission and medium–small capital and consumption.

Let us consider the economic state [0.9404, 0.2215, 0.8462] — Case (6). The corresponding emission level is 0.2069. This point is in the upper right part of each slice. We interpret this state as characterizing an economy with (rather) high capital, high consumption, and lax emission control.

This state is viable, that is, a strategy for u exists, and hence z, which can keep the entire economic evolution, hence for $t \in [0, \infty)$, inside the kernel $\mathcal{V}_F(K) \subset K$, such that $X \leq 0.275$. Figure 3 presents this evolution in the 2D (k, c) and (k, z) slices marked by solid black lines. Figure 4 shows the corresponding time profiles as depicted by the black solid lines in each panel. The evolution is rather short: it takes nine periods to reach a steady state.

A possible method of maintaining this evolution in K is to relax the emission control even further, that is, up to a value of $z = 0.9156$. This process enables the economy to preserve consumption and only marginally decrease capital. Failure to do so could lead the emission control to drain the capital and the economy would leave K before a steady state would have been achieved.

Many substantially high capital states with the same consumption and emission control are nonviable. Consider the state [1.1173, 0.2215, 0.8462] — Case (7). Figure 3 shows this point located to the right from the previous point (Case (6)) in each panel, and represented as a red star. The initial emission is higher than before ($X = 0.2217$ *vs.* $X = 0.2069$) but remains within the upper bound of $\bar{X} = 0.275$. However, the right panel displays this state as nonviable because the evolution of the economy from this state fails to stabilize before it hits one of the constraints of K. Figure 5 presents this case as depicted by the red solid line (the first from the top in the first five panels), with $c(93) < 0.01$ and $k(93) > 1.25$. Initially, emission increases. In period 4, emission control is tightened ($u < 0$). However, increasing emission control (smaller z) that could stimulate consumption fails because capital (actually, $k^{0.6}$)

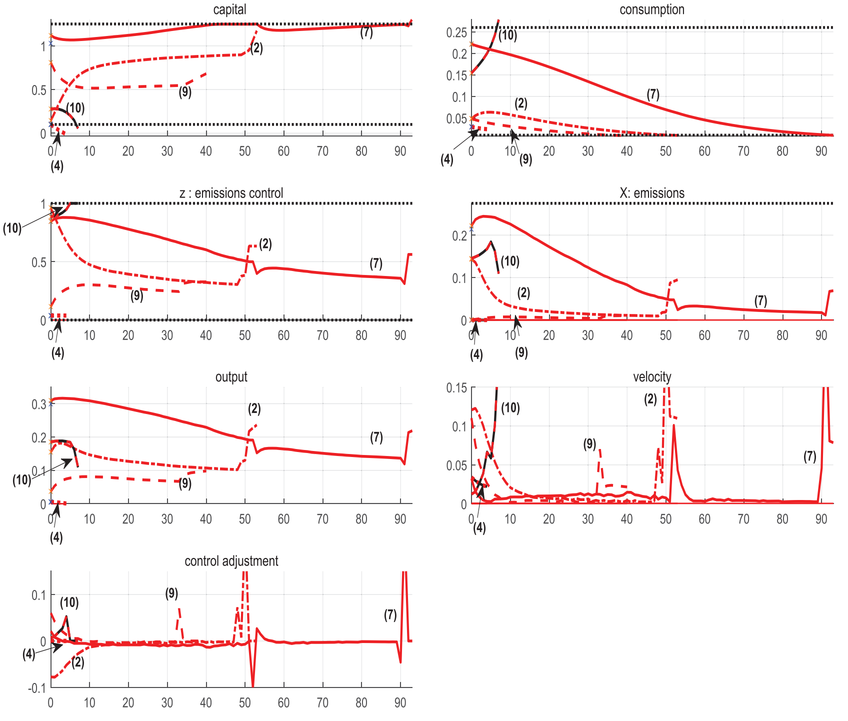

Figure 5
Time Profiles of Non-Viable Evolutions

grows faster than z. Consequently, consumption gradually decreases and violates its lower bound at period 93.

Another point with large capital and minimum emission control (see *e.g.*, [1.1173, 0.2215, 0.9615] — Case (8)) — marked as a red "o" in Figure 3 (right panel), fails the emission constraint. Accordingly, $X = (0.35)\cdot(1.1173)^{0.4}\cdot(0.9615)^3 = 0.3252$.

Lastly, we note that nearly any level of capital *may* be viable (Figure 3, right panel). However, after looking at the large nonviable areas in the left panel in Figure 3, we realize that several combinations of capital and consumption are nonviable. We now explain what happens to an economy rich in capital if consumption is not "aligned" with capital.[11]

[11] Aligning with capital means consumption remains in a viability corridor. Compare footnote 7.

Consider the state [0.8077, 0.0485, 0.1154] — Case (9). Figures 3 and 5 depict the evolution from this state as red dashed lines. Consumption slides to below the lower limit within 40 periods. The system's dynamics is such that the emission control relaxation (hence increasing z) is sufficient to maintain a steady (or, eventually, growing) capital. However, this growth of z is insufficient to prevent consumption from vanishing.

Consumption "dominates" capital on the other side of the k, c corridor in the left panel in Figure 3. Consider the state [0.2769, 0.1542, 0.8846] — Case (10). Figures 3 and 5 represent evolution from this state as red–black dashed lines. Capital diminishes to below its lower limit within 7 periods. Consumption increases and violates its upper bound. The system's dynamics is such that the emission control relaxation facilitates the growth of consumption. However, capital cannot sustain this growth and dissipates.

c) Qualitative Behavior

Consider the qualitative behavior of the system's dynamics for initially high, medium high, and medium values of z. They correspond to very lax, relaxed, and active levels of abatement, respectively (low – "ambitious" – values of z will be discussed thereafter).

Figure 6 shows the time profiles of the bunches of viable evolutions. The blue (solid), green (dotted), and red (large dotted) lines represent the bunches of the initially high, medium high, and medium z values, respectively.

For viability, high levels of z (poor abatement, blue lines) require negative emission-control adjustments. The case is evident in the last panel in Figure 6 (control adjustment), see the blue lines below 0. Consequently, emission X decreases, increases, or remains constant. Whether emission X increases or decreases depends on the initial levels of c and k.

The medium values of z, represented by red lines, correspond to active abatement. For viability, the states that include medium values of z require positive adjustments. As a result, emission levels will increase or stay steady (see the red lines in the emissions panel).

The medium-high levels of z (green lines) do not attract any specific pattern of emission-control adjustments.

We consider the qualitative behavior of the system's dynamics for very low and medium-low values of z. These values correspond to

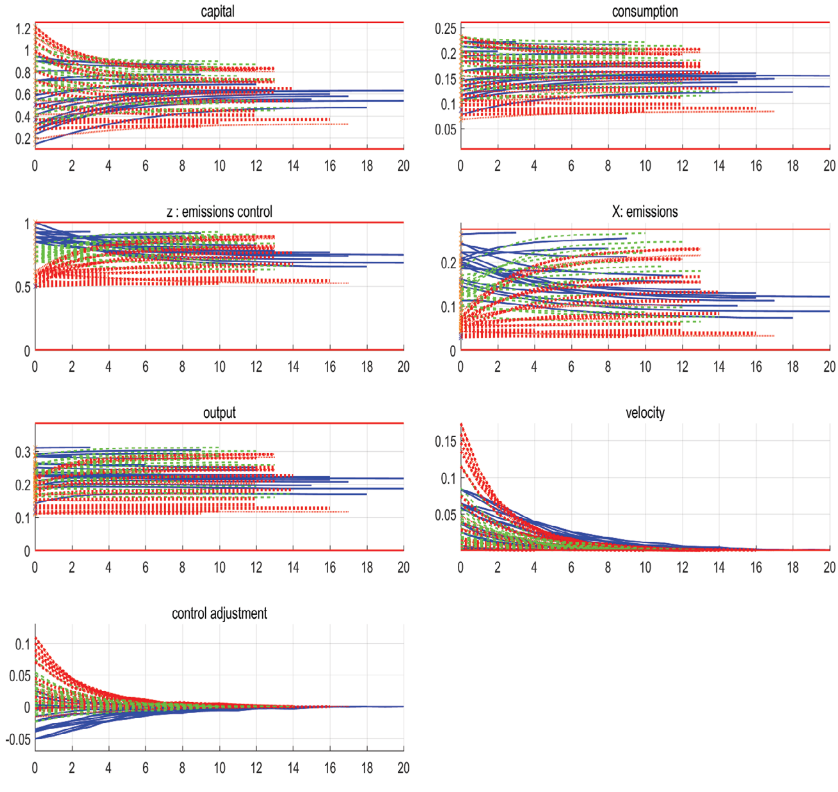

FIGURE 6
TIME PROFILES OF BUNCHES OF VIABLE EVOLUTIONS FOR LOW AND MEDIUM-LOW ABATEMENT

"ambitious" abatement. The bunches of black (dashed–dotted) and red (solid) lines in Figure 7 represent, respectively, very strict and strict initial levels of abatement.

The last panel in Figure 7 shows that (for viability) the low levels of z (strict abatement, see both colors) require positive emission-control adjustments. Accordingly, emission X increases or remains constant.

The strict and very strict abatement programs (*i.e.*, low z) differ between themselves. Mainly, very strict programs stabilize at low capital, low consumption, low output and very low emission. The less strict abatement programs can help the economy to stabilize at low-to-medium high capital, low-to-high consumption, low-to-medium output, and low-to-medium emission.

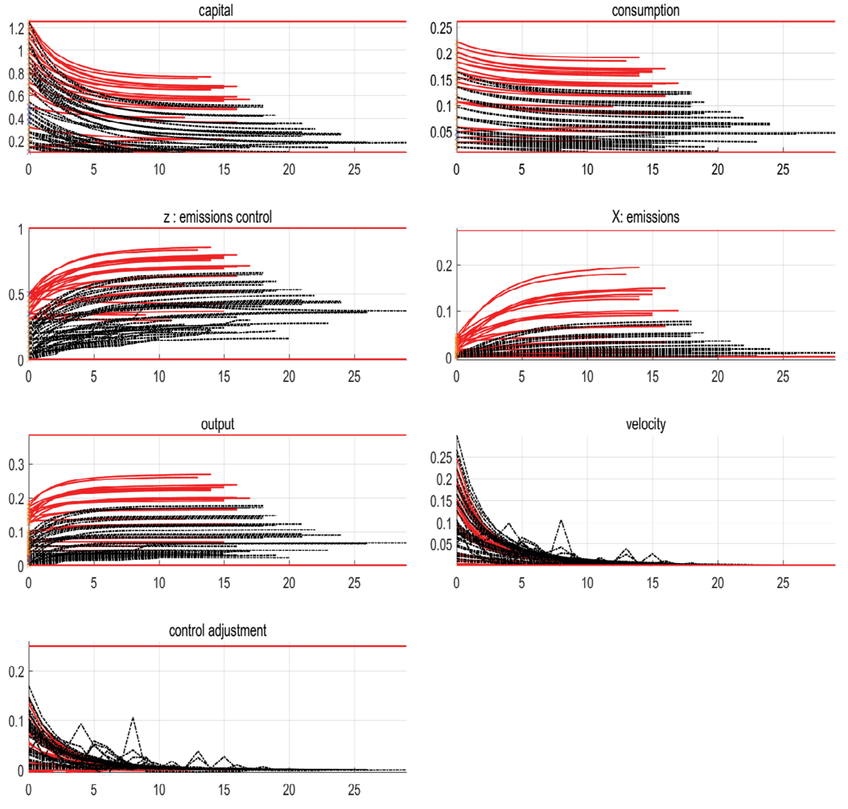

Figure 7

Time Profiles of Bunches of Viable Evolutions for Strict Abatement

Figures 6 and 7 show that, unless the initial capital levels are very low, emission controls reduce capital and consumption.

Furthermore, our analysis led us to propose that the general directions of economic evolutions can be anticipated when economies are subjected to emission control. For example, very strong emission controls tend to exhaust capital and initial lax emission controls result in medium range capital and consumption, among others.

d) Transition from Unconstrained to Constrained Emission

Our results enable us to analyze strategies that can lead an economy with unconstrained emission to that with constrained emission.

Figure 8 shows the economic time profiles of three evolutions with

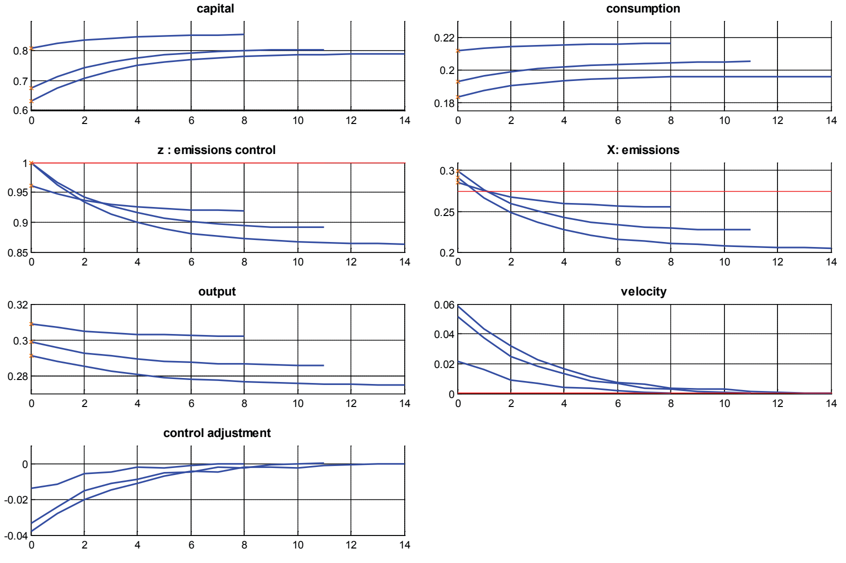

Figure 8

Time Profiles for Transitions from Unconstrained to Constrained Emission

initial emission levels X above the upper bound on emission $\bar{X} = 0.275$. Hence, these evolutions originate outside kernel $\mathcal{V}_F(K)$ (Figure 9) and also outside the constraint set K. However, for each evolution, the capital–consumption combinations are reasonably aligned and contained in the viability kernel $\mathcal{V}_F(K_u)$ (gray boulder). Each evolution, through gradual adjustment of z, eventually moves into the kernel $\mathcal{V}_F(K) \subset K$ and stabilizes.

Thus, these three evolutions are viable in K_u and each reaches a viable point in $\mathcal{V}_F(K) \subset K$ through emission control adjustments. Figure 9 illustrates these transitions, represented by the solid blue lines, which go from the gray to the purple boulder.

When capital and consumption are high, gradually lowering z will diminish emission and lead the economy to the emission-constrained viability kernel. Lower levels of capital and consumption (two bottom lines in the top panels in Figure 8) require rapid changes of emission control to reach a viable point in the middle ranges of capital and consumption.

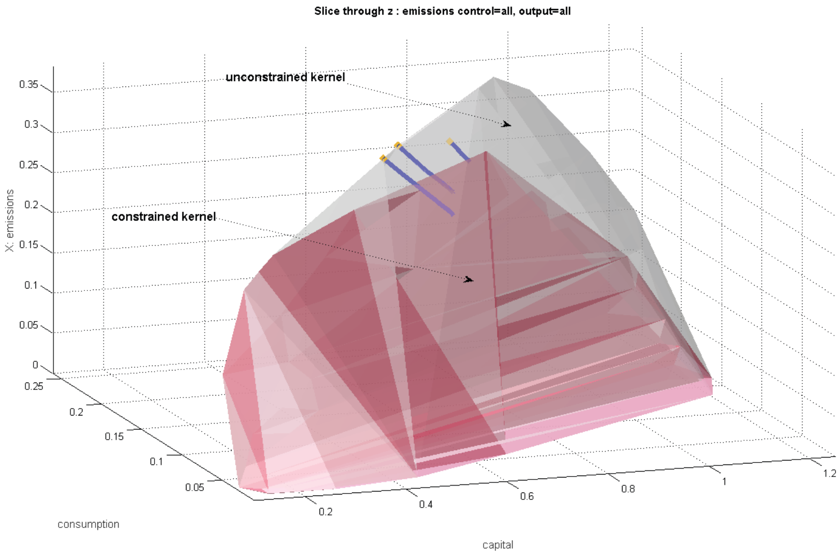

Figure 9

Kernels for the Constrained and Unconstrained Emission with Transition Paths

V. Concluding Remarks

We presented a computational method based on viability theory to discover consumption choices that are compatible with state variables in a polluting economy. The compatibility means that viable consumption and capital choices can be controlled to a nearly steady state by a smooth emission-control adjustment policy.

Among other findings, we report that, as long as an initial combination of consumption, capital, and emission control is inside the viability kernel of the *unconstrained* emission problem, there may exist smooth evolutions of emission control that can reach viable levels of capital, consumption, emission control, and the resulting emission.

Our study can be used as a basic framework to devise a sustainable public policy toward abating ultrafine dust and maintaining economic livelihood.

Our calibrated model enables us to formulate some policy advice. The most important advice concerns the initial capital–consumption combinations. An economy that has a “wrong” capital–consumption ratio must correct it (*e.g.*, by asking for international aid) before engaging in emission control by its own means.

Positive advice, based on the viability kernel, can be formulated on what emission-control adjustment strategy should be applied to comply with the emission constraint and to keep the economy inside the desired constraint set of the state space.

We propose that viability theory is a beneficial framework for analyzing sustainability when the policymaker is facing trade-offs among multiple objectives. This study applied this framework to the environment *vs.* growth issue. Extending the model to include other welfare-enhancing variables, such as human capital, inequality, and social cohesion, is one of our ongoing research projects.

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