Money and Capital Adjustment: Revisiting the Role of Money for Production

Jae Eun Song

This study presents a new monetarist model, in which monetary liquidity is essential for capital adjustment, to analyze the effects of money growth on the production side. Results from revisiting the classic issue of money and capital with this model highlight the role of the diminishing rate of technical substitution between money and capital in the adjustment. When the substitutability between money and capital is sufficiently high, the positive Tobin effect of inflation on the aggregate capital stock and output can dominate the negative price distorting effect of inflation. This case will likely occur when search frictions in the capital market are severe.

Keywords: Money growth, Inflation, Capital stock, Market friction, Liquidity

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I. Introduction

The long-run relationship between money growth (or anticipated inflation) and capital accumulation is a significant but controversial issue in macroeconomics. Tobin (1965) initially argues that the

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relationship is positive. Inflation cuts down the rate of real return to money, thereby possibly inducing agents to switch from holding money to investing capital. However, in addition to this substitution effect, Stockman (1981) explains that the negative wealth effect of inflation must be considered. Inflation also increases the cost of holding money for future consumption and investment.[1]

Traditional studies on this issue provide well-known and diverse conclusions. In a standard money-in-the-utility-function model (*e.g.*, Sidrauski 1967), money is superneutral in the long run because the two effects cancel each other. However, in a standard cash-in-advance model, which highlights the negative real balance effect, long-run inflation is associated with reduced aggregate capital stock. Athough previous studies based on these kinds of models have significantly contributed in analyzing the issue, there is skepticism regarding the reduced-form approach of these studies.[2]

Given the developments in a search-theoretic approach that emphasizes microfoundations for trading with frictions, attempts have been made to study the capital-theoretic issue through a new lens. Shi (1999) shows that a positive extensive effect of money growth can dominate its conventional negative intensive effects.[3] Molico, and Zhang (2005) present numerical results wherein a distributional effect of lump-sum transfers of money leads a positive long-run relationship between moderate inflation and capital accumulation. Lagos, and Wright (2005) develop a two-sector framework that integrates the competitive and search-theoretic markets to incorporate the frictions of microfounded theory into practical models for policy analysis. Aruoba, Waller, and Wright (2011) adopt this framework, revisit the issue, and show that

[1] Together with elastic labor supply, inflation also distorts the consumption-leisure choice of households as discussed in Cooley, and Hansen (1989). Therefore, a reduction in the real money balance due to inflation adversely affects capital accumulation through another channel as long as labor and capital are technical complements in production.

[2] They introduce a special function of money as if money has it per se without explicitly modeling frictions that derive it as an outcome. Wallace (2001) discusses hidden inconsistencies in these reduced-form models, in which the real money balance is assumed to be productive.

[3] A high cost of holding money by inflation makes households spend money rapidly. Increasing the number of buyers in the decentralized goods market increases the frequency of successful trades in it and facilitates investment.

different pricing mechanisms in the search-theoretic market contribute significant quantitative differences in the effects of money growth.[4]

The current study analyzes a link that channels monetary policy to capital formation, focusing on demands for monetary liquidity in the production side of an economy. These demands are associated with the service of money to firms' transactions for capital adjustment. Even though most macrostudies focus on liquidity demands for consumption, some researches focus on the production side. Money-in-the-production-function models employ a reduced-form approach, and based on these models, various empirical studies following Sinai, and Stokes (1972) provide evidence for the positive relation between real money balance and aggregate production.[5]

Nevertheless, the frictions that link individual firms' liquidity problem with the public supply of monetary liquidity is not explicitly modeled in mainstream macrostudies. By contrast, Holmström, and Tirole (1998) address a distinguished role for government-supplied liquidity on the production side. Kiyotaki, and Moore (2001) explicitly model frictions that motivate liquid assets to function in smooth investments. In their business cycle model, money is essential (as the most liquid asset) for the effective allocation of resources. However, this research focuses on the fluctuations by productivity and liquidity shocks and the role of monetary stabilization policy rather than the classical issue of long-run inflation.[6]

[4] A survey of new monetarism is included in Williamson, and Wright (2010). Some researches in the literature analyze how money and capital compete as media of exchange. By contrast, others consider the situation in which capital cannot be used as a means of payment to focus on how the pricing mechanism in the search-theoretic market channels the effect of money growth to capital accumulation. There also exist many works that lay out new monetarist models of ideas in earlier monetarist or Keynesian traditions (*e.g.*, Kim and Lee 2012).

[5] Lotti, and Marcucci (2007) show that the demand for monetary liquidity by US non-financial firms is significant using a firm-level money-in-the-production-function model. In a standard money-in-the-production-function model, inflation cuts down the real balances of money and usually has a negative effect on the aggregate capital stock in the long run. However, this prediction is critically affected by the specification of the manner in which money or its transaction service is incorporated into the production function.

[6] In addition, assuming that all markets are competitive, their model is not explicit about the frictions that make a medium of exchange essential. Hence, the transition role of money is not explicitly determined in the model as in

In this paper, to determine the transaction role of money on the production side, a model is contructed in which agents want to trade capital due to idiosyncratic uncertainty in its factor productivity. Agents in the model economy must prepare physical capital before they observe the realization of its stochastic productivity in their own production. Hence, a market in which agents trade capital after the shocks are realized is essential for the economy. In addition, the Lagos-Wright framework is employed assuming that this capital market is decentralized and has frictions that require monetary trade, and that all other markets are competitive. In this case, capital cannot be used as a means of payment in trading itself. Thus, capital in this environment cannot compete with money as a medium of exchange although it is storable.[7]

A key feature of the proposed model is speculative money holding, that is, agents hold money to use an additional investment opportunity that may arise later on. In addition, the installation of capital prepared by others for immediate use requires paying adjustment costs. As is commonly assumed, the unit cost of adjustment positively depends on the percentage rate of increase in capital stock. This situation implies that capital purchased by using money and capital installed before productivity shocks can be considered two factors of production for additional capital input. In case that the bargaining power of buyers in the capital market is sufficiently strong, the production for additional capital shows the diminishing rate of technical substitution between money and capital. Thus inflation leads to the portfolio substitution out of money to capital because it raises the cost of holding money.

However, the portfolio substitution out of money into capital increases the investment demand, and households reduce consumption and increase work to save additional capital. This situation demonstrates a price distorting effect of the inflation tax, which is distinct from the standard real money balance effect. However, if money and capital are strong substitutes in the production for additional capital input, and hence the direct effect of the portfolio substitution is sufficiently large,

overlapping generations models.

[7] One issue in monetary theory is to endogenously determine the objects that serve the essential role of medium of exchange. The model in the current study highlights the fact that a commodity cannot serve this role in trading itself even if it qualifies for the role in trading other commodities.

then inflation increases the aggregate capital and output in equilibrium. This case is likely to occur when search frictions in the capital market are severe.

The rest of this paper is organized as follows. Section II describes the model. Section III defines the equilibrium and characterizes it. Section IV investigates the effects of the anticipated inflation on the steady state. Section V presents the conclusion. The Appendix provides all proofs of the lemmas and propositions.

II. Model

A. Environment

Time is indexed by $t = 0, 1, 2, \ldots, \infty$. A continuum of infinitely lived households with total mass equal to 1 exists. Each household consists of a worker and an entrepreneur, and they share earnings and maximize the expected lifetime utility of the household.[8] Four objects are traded in this economy: a perishable general good, physical capital, labor, and fiat money.

Workers are homogenous, and each worker has $\bar{h} \in \mathbb{R}_{++} \equiv (0, \infty)$ hours of time for each period. In each period, every worker derives utility $u(c) + l$ from consuming $c \in \mathbb{R}_{+}$ units of the general good and enjoying $l \in \mathbb{R}_{+}$ hours of leisure.[9] The quantity of the net labor supplied by a worker is $\bar{h} - 1$, and $l > \bar{h}$ means that the worker consumes $l - \bar{h}$ hours of housework services supplied by other workers.[10] Assume that u is twice continuously differentiable, $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. The discount factor across periods is $\beta \in (0, 1)$.

[8] Two members are assumed to exist in a household to introduce a competitive labor market in a simple manner. Alternatively one can assume that each household has a single agent who owns a profit-maximizing firm. The key point is that these firms must be independent decision units in the labor market, although they obtain investment funds from their owners.

[9] The marginal utility of leisure is a constant, and its size is merely a matter of the measurement unit; hence, it can be normalized to one without loss of generality. See Wallace (2002) for insights into the role of quasi-linear utility in the Lagos–Wright framework.

[10] The introduction of housework service or transferable leisure ensures the interiority of optimal choices at the individual level. This step avoids the complexity caused by a heterogeneous portfolio distribution due to diverse non-labor incomes.

Entrepreneurs neither consume nor supply labor. However, they can produce a general good using their capital and employing workers in the other households. The production function for a type $i \in \{L, H\}$ entrepreneur is $F(z_i\kappa, h)$, where $z_i\kappa \in \mathbb{R}_+$ is the capital stock in terms of effective units and $h \in \mathbb{R}_+$ is the labor employment. Assume that F is twice continuously differentiable and is the standard neoclassical production function. $\kappa \in \mathbb{R}_+$ represents the deterministic base of effective capital, and an entrepreneur's type indexes an idiosyncratic shock $z_H > z_L > 0$ on its factor productivity in his or her current-period production. Let $z \equiv z_H / z_L$, and normalize z_L to 1. The shocks are randomly drawn at the beginning of each period, and their distributions are independent and identical over time. The probability that an entrepreneur draws type H is $\pi \in (0, 1)$.

Similar to standard models, capital available as input to production must be formed at least one period prior. Entrepreneurs freely transform one unit of the general good into one unit of physical capital, which serves their own production as one deterministic base unit of effective capital from the next period. However, by paying the adjustment costs, entrepreneurs can also use the capital made by others for their production. From $q \in \mathbb{R}_+$ units of the physical capital delivered from others, an entrepreneur who holds $k \in \mathbb{R}_{++}$ units of physical capital that he prepared obtains $f(q, k) \equiv q - k\Gamma(q/k)$ units of the deterministic base, that is, $\kappa = k + f(q, k)$. Γ is assumed to be twice continuously differentiable, $\Gamma > 0$, $\Gamma'' > 0$, $\Gamma(0) = 0$, $\Gamma'(0) = 0$, and $\Gamma'(\infty) = 0$. In addition, the *installation function* f is defined on $\mathbb{R}_+^2$ by letting $f(q, 0) \equiv 0$ for all $q \in \mathbb{R}_+$. A capital stock in terms of effective units depreciates at a rate of $\delta \in [0, 1]$ in a period and can be converted back into the same quantity of the general good.

To explicitly determine the entrepreneurs' demand for liquidity, revenues from production are assumed to be unavailable for purchasing capital utilizable in the same period. This restriction is modeled by the sectoral structure of Lagos, and Wright (2005). Each period has two subperiods called day and night, and it begins at night. Production for the general good in each period starts at the beginning of the day, and capital should be delivered by the end of the preceding night to be utilized for the production (Figure 1).

Money is exogenously supplied and perfectly storable. Total money stock increases between day $t-1$ and day t at a gross rate $\tau_t \in \mathbb{R}_{++}$. M_t indicates total money stock at night t. Given the initial $M_0 \in \mathbb{R}_{++}$, at the

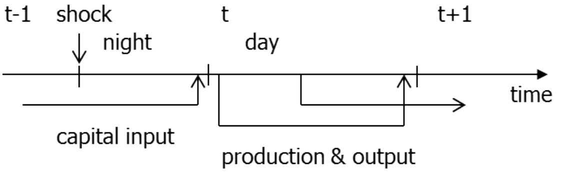

Figure 1
Day and Night

beginning of each day t, $(\tau_t - 1)M_t$ units of new money are injected to each household in the form of lump-sum transfer, or taxes if $\tau_t < 1$.

Every day, all the agents have an access to a Walrasian auction wherein they trade the general good, labor, and money. Thus, as in standard general equilibrium theory, daytime trades achieve a competitive allocation without using a medium of exchange.[11] This means we can think of three competitive markets for each object, which clear simultaneously at a pair of relative prices. P_t and W_t denote the nominal price of the general good and nominal wage per hour in day t, respectively.

At night, entrepreneurs trade capital and money after the realization of the productivity shocks. This market is decentralized with bilateral random matching. Each entrepreneur at night initially chooses whether he or she buys capital or sells it, and searches for his or her trading partner in the market. Under the standard constant returns to scale matching technology, if the seller-buyer ratio is $n \in \bar{\mathbb{R}}_+ \equiv [0, \infty]$, then a buyer is matched with a seller with probability $\alpha(n)$, and a seller is matched with a buyer with probability $\alpha(n)/n$.[12] Entrepreneurs are

[11] The daytime economy is the centralized sector in the Lagos–Wright framework. However, Williamson, and Wright (2010) clarify that the centralization does not mean that everyone is simultaneously in the same place. Such situation threatens not only the essentiality of money in the framework but also the noncooperative foundation of any competitive market model. Anonymity can be considered a condition that must be satisfied in modeling a competitive market (see Osborne, and Rubinstein 1990).

[12] The standard matching technology describes search frictions, which make some buyers and sellers simultaneously left unmatched. However, search frictions are unnecessary for the results in the present study. If assuming that the capital market has a search friction is undesirable, then $\alpha(n) \equiv \min\{1, n\}$ can be used.

anonymous at night. Thus, trading in any match must be quid pro quo, and money must be used.[13] Moreover, capital is exchanged only for fiat money under the assumption that any claim issued by the households can be costlessly counterfeited.[14] Each party in a match observes the partner's type and portfolio, and the terms of trade are given by the generalized Nash bargaining solution with a buyer's bargaining power $\theta \in (0, 1]$.

B. Value Functions and Decisions

Hereafter the quantity of money in period t is expressed as a fraction of the total money stock at night t. Thus, the unit price of money indicates the real money balance. In addition, current leisure is used as the numeraire for daytime pricing, and thus prices are expressed in terms of utils. The price of the general good in day t is denoted by $p_t \equiv P_t/W_t$ and that of money by $\phi_t \equiv M_t/W_t$. In terms of the general good, the real wage per hour and real money balance in the day are $1/p_t$ and ϕ_t/p_t respectively. In equilibrium, $p_t \in \mathbb{R}_{++}$ for all t. This study focuses on monetary equilibrium, in which $\phi_t \in \mathbb{R}_{++}$ for all t.

$G_t \in \Delta(\mathbb{R}^2_+)$ represents the portfolio distribution at the time the capital productivity shocks at night t arrive, where $\Delta(\mathbb{R}^2_+)$ is the set of probability measures on the Borel subsets of $\mathbb{R}^2_+$. For any Borel set $A \subseteq \mathbb{R}^2_+$, $G_t(A)$ is the proportion of entrepreneurs who hold k units of capital and m units of money, such that $(k, m) \in A$ at that time. V_t is the value function, such that $V_t\,(k, m)$ indicates the expected lifetime utility of a household that enters period t with k units of physical capital and m units of

[13] See Kocherlakota (1998) and Williamson, and Wright (2010) with the references therein for discussions on the essentiality of money and the role of *anonymity*. For the essentiality, only a segment of buyers at night must be anonymous to their trading partners. Thus, credit trades of capital can be easily incorporated into the model by introducing exogenous heterogeneity among the trading matches. However, small gains are obtained from this extension theoretically because it leads to dichotomy between the money and credit sectors. Nosal, and Rocheteau (2011) note that the framework to introduce costs of using credits should be seriously considered for the extension.

[14] A challenging question is what object emerges as money among storable assets. The current study does not aim to obtain an answer; *recognizability* is simply assumed to generate frictions by which only outside money serves as a medium of exchange. See Lester, Postlewaite, and Wright (2011) and Rocheteau (2011) with the references therein for discussions on the role of recognizability.

money. In each period t, individual agents take G_t, p_t, ϕ_t, V_{t+1}, and τ_t as given. They also take as given the rule that determines the terms of trade in the capital market.

Consider a household with $\mathfrak{K}$ effective units of capital and m units of money at the end of night t; $W_t(\mathfrak{K}, m)$ indicates its expected lifetime utility. Let $\hbar$: $\mathbb{R}_{++} \mapsto \mathbb{R}_{++}$ be the *labor demand function* implied by $pF_2(1, \hbar(p)) = 1$. For any real wage per hour $1/p_t \in \mathbb{R}_{++}$, each entrepreneur employs $\hbar(p_t)$ hours of labor per effective unit of capital, and each earns real profit $r_t \equiv F_1(1/(\hbar(p_t), 1)) + 1 - \delta$ per effective unit of capital. Thus,

$$W_t(\mathfrak{K}, m) \equiv \max_{c,\, l,\, k',\, m'} \{u(c_t) + l + \beta V_{t+1}(k', m')\}$$

subject to the non-negativity constraints and

$$p_t c + l + p_t k' + \tau_t \phi_t m' \leq \bar{h} + p_t r_t \mathfrak{K} + \phi_t (m + \tau_t - 1).$$

The decision rules in any particular period are each defined as a function of individual states $\mathfrak{K}$ and m. However, the budget constraint must bind; hence, l can be eliminated. Accordingly, the following equation is achieved:

$$\begin{aligned} W_t(\mathfrak{K}, m) = p_t r_t \mathfrak{K} + \phi_t m + \phi_t(\tau_t - 1) + \bar{h} + \max_c \{u(c) - p_t c\} \\ + \max_{k',\, m'} \{- p_t k' - \tau_t \phi_t m' + \beta V_{t+1}(k', m')\}. \end{aligned}$$

c: $\mathbb{R}_{++} \mapsto \mathbb{R}_{++}$ is the *consumption demand function* implied by $u'(c(p)) = p$. In each period t, all households consume $c(p_t)$ units of the general good. In addition, each household chooses a portfolio (k_t', m_t') independently on its individual states. However, if an optimal choice is not unique, then households may choose different portfolios. For expositional brevity, this study assumes that in each period t, all households use the same behavioral strategy $\mathcal{G}_t \in \Delta(\mathbb{R}_+^2)$ for portfolio choice. The portfolio that is actually chosen by a household using a behavioral strategy is randomly drawn from the distribution specified by the strategy.

Consider a type i buyer with portfolio (k_b, m_b) and a type j seller with portfolio (k_s, m_s) in the capital market at night t. If they are matched and exchange q units of capital with d units of money, then the buyer and seller's gains from the trade are as follows:

$$W_t(z_i[k_b + f(q, k_b)], m_b - d) - W_t(z_i k_b, m_b) = p_t r_t[z_i f(q, k_b) - \psi_t d],$$
$$W_t(z_j[k_s - q], m_s + d) - W_t(z_j k_s, m_s) = p_t r_t[\psi_t d - z_j q],$$

where $\psi_t \equiv r_t^{-1} \phi_t/p_t$ is the relative value of money compared with low productivity capital in day t. The total surplus $p_t r_t[z_i f(q, k_b) - z_j q]$ is positive only if $i = H$, $j = L$, and $k_b > 0$. If this is the case, with $\ell_t \equiv \psi_t d$, the terms of trade solve

$$\max_{q,\, \ell_t}[zf(q, k_b) - \ell_t]^{\theta}\,[\ell_t - q]^{1-\theta} \tag{1}$$

subject to $0 \leq q \leq k_s$ and $0 \leq \ell_t \leq \psi_t m_b$. Let q, ℓ: $\mathbb{R}_+^3 \mapsto \mathbb{R}_+$ be the pair that constitutes the solution function: for any given $(\psi_t m_b, k_s, k_b) \in \mathbb{R}_+^3$, $(q(\psi_t m_b, k_s, k_b), \ell(\psi_t m_b, k_s, k_b))$ solves the maximization problem. In the match, $q(\psi_t m_b, k_s, k_b)$ units of capital are exchanged for $\psi_t^{-1}\ell(\psi_t m_b, k_s, k_b)$ units of money. Define B, S: $\mathbb{R}_+^3 \mapsto \mathbb{R}_+$ by

$$B(x, \tilde{k}, k) \equiv zf(q(x, \tilde{k}, k), k) - \ell(x, \tilde{k}, k),\ S(x, k, \tilde{k}) \equiv \ell(x, k, \tilde{k}) - q(x, k, \tilde{k}).$$

The surplus for the buyer is $p_t r_t B(\psi_t m_b, k_s, k_b)$ and that for the seller is $p_t r_t S(\psi_t m_b, k_s, k_b)$.

In monetary equilibrium, the probability of obtaining surplus from spending money must be positive in each night. Each night, all type H entrepreneurs become capital buyers. In addition, all type L entrepreneurs become sellers each night under the assumption that any type L chooses to sell capital if indifferent. Hence, in any monetary equilibrium, the seller-buyer ratio in the capital market is $n_t = (1 - \pi)/\pi$ for all t. Let $\varsigma \equiv \pi\alpha((1 - \pi)/\pi)$ denote the total mass of matches in the capital market each night, and for any $G \in \Delta(\mathbb{R}_+^2)$, define $E_G : \mathbb{R}_+^2 \times \mathbb{R}_{++} \mapsto \mathbb{R}_+$ by

$$E_G(m, k, \psi) \equiv \int [B(\psi m, \tilde{k}, k) + S(\psi \tilde{m}, k, \tilde{k})]G(\mathrm{d}(\tilde{k}, \tilde{m}))$$

For an entrepreneur with k units of capital and m units of money at the beginning of night t, the expected surplus from trading capital at night is $\varsigma p_t r_t E_{G_t}(m, k, \psi_t)$. Thus,

$$\begin{aligned} V_t(k, m) = {} & \varsigma p_t r_t E_{G_t}(m, k, \psi_t) + p_t r_t \bar{z}k + \phi_t m + \phi_t(\tau_t - 1) + \bar{h} + c(p_t) \\ & + \max_{\mathcal{G}} \int [-\, p_t k' - \tau_t \phi_t m' + \beta V_{t+1}(k', m')]\mathcal{G}(\mathrm{d}(k', m')), \end{aligned} \tag{2}$$

where $\bar{z} \equiv \pi z + 1 - \pi$ is the expected capital productivity shock.

III. Equilibrium

First charcterize the market clearing conditions for the daytime competitive markets. Given $G \in \Delta(\mathbb{R}_+^2)$, let G_k, G_m be the marginal measures defined by $G_k(A) \equiv G(A \times \mathbb{R}_+)$, $G_m(A) \equiv G(\mathbb{R}_+ \times A)$ for all Borel sets $A \subseteq \mathbb{R}_+$. G_{tk}, G_{tm} represent the marginal distributions of capital and money holdings at the beginning of night t. The aggregate capital stock at that time is given by $K_t \equiv \int k G_{tk}(\mathrm{d}k)$. For any $G \in \Delta(\mathbb{R}_+^2)$, define $T_G : \mathbb{R}_{++} \mapsto \mathbb{R}_+$ by

$$T_G(\psi) \equiv \int\int [zf(q(\psi m, \tilde{k}, k), k) - q(\psi m, \tilde{k}, k)]\ G_k(\mathrm{d}\tilde{k})\ G(\mathrm{d}(k, m)).$$

All trades at night t increase the aggregate capital stock by $\varsigma T_{G_t}(\psi_t)$ in effective units. Thus, the goods market clearing in day t requires

$$c(p_t) + \int k' \mathcal{G}_{tk}(\mathrm{d}k') = [\bar{z}K_t + \varsigma T_{G_t}(\psi_t)][F(1, \hbar(p_t)) + 1 - \delta]. \tag{3}$$

The money market clearing in day t requires $\int m' \mathcal{G}_{tm}(\mathrm{d}m') = 1$. Each day, if the goods and money markets clear, then the labor market clears by Walras' law.

Definition 1. An equilibrium consists of a sequence $\{V_t, \mathcal{G}_t, G_t, p_t, \phi_t\}_{t=0}^{\infty}$ and a set of time-invariant functions $\{c, \hbar, q, \ell\}$, where for every t, V_t is the value function, $\mathcal{G}_t$ is the behavioral strategy for portfolio choice, G_t is the distribution of capital and money holdings, p_t, ϕ_t are the daytime prices, c, $\hbar$ are the daytime decision rules, and (q, ℓ) is the rule for the terms of capital trade. The equilibrium conditions are as follows:

(i) Given $\{G_t, p_t, \phi_t\}_{t=0}^{\infty}$ and $\{c, \hbar, q, d\}$, $\{V_t, \mathcal{G}_t\}_{t=0}^{\infty}$ satisfies (2);
(ii) c, $\hbar$ are the consumption and labor demand functions;
(iii) (q, ℓ) is the solution function of the problem (1);
(iv) the daytime markets clear at $p_t, \phi_t \in \mathbb{R}_{++}$ for all t;
(v) $G_{t+1} = \mathcal{G}_t$ for all t; and G_0 with $K_0 \in \mathbb{R}_{++}$ is given.

Begin to characterize the equilibria by deriving the rule for the terms of capital trade. Let $\mathfrak{F}(\cdot) \equiv f(\cdot, 1)$ denote the percentage rate installation function. Then $\mathfrak{F}(0) = 0$. In addition, from $\mathfrak{F}(\eta) = \eta - \Gamma(\eta)/\eta$ for all $\eta \in \mathbb{R}_{++}$, we obtain $\mathfrak{F}' > 0$, $\mathfrak{F}'' < 0$, $\mathfrak{F}'(0) = 1$, and $\mathfrak{F}'(\infty) = 0$. Given that $z > 1$, there exists $\eta^* \in \mathbb{R}_{++}$ such that $z\mathfrak{F}'(\eta^*) = 1$. Define $\hat{\mu}, \breve{\mu} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by

$$\hat{\mu}(\eta) \equiv z[\theta\eta\mathfrak{F}'(\eta) + (1-\theta)\mathfrak{F}(\eta)]\,[\theta z\mathfrak{F}'(\eta) + 1 - \theta]^{-1},$$
$$\breve{\mu}(\eta) \equiv \theta\eta + (1-\theta)z\mathfrak{F}(\eta).$$

Then $\breve{\mu}' > 0$, $\hat{\mu}(0) = \breve{\mu}(0) = 0$, and $\hat{\mu}(\eta^*) = \breve{\mu}(\eta^*)$. After a simple calculus, $\hat{\mu}' > 0$ is obtained. Let $\mu^* \equiv \hat{\mu}(\eta^*)$ and $\hat{\eta} \equiv \hat{\mu}^{-1}$ for notational brevity.

Lemma 1. *If* (q, ℓ) *is the solution function of the problem (1), for any* $k_b \in \mathbb{R}_{++}$,

$$q, \ell(x, k_s, k_b) = \begin{cases} \eta^* k_b & \mu^* k_b & \text{Case I} \\ \hat{\eta}(x/k_b)k_b & x & \text{Case II} \\ k_s & \breve{\mu}(k_s/k_b)k_b & \text{Case III} \\ k_s & x & \text{Case IV} \end{cases}$$

Case I	$x \geq \mu^* k_b$	$k_s \geq \eta^* k_b$
Case II	$x < \mu^* k_b$	$k_s \geq \hat{\eta}(x/k_b)k_b$
Case III	$x \geq \breve{\mu}(k_s/k_b)k_b$	$k_s < \eta^* k_b$
Case IV	otherwise	

and $q(x, k_s, 0) = \ell(x, k_s, 0) = 0$ *for all* $(x, k_s) \in \mathbb{R}^2_+$.

The socially optimal quantity of capital transfer is $\eta^* k_b$. However, at night t, the optimal quantity traded under bargaining requires real money balance $r_t\mu^* k_b$ in terms of the general good. The quantity traded is constrained by the buyer's liquidity and seller's inventory. The liquidity constraint binds only in Case II, whereas the inventory constraint binds only in Case III. Both constraints bind in Case IV.

Thereafter, individual portfolio optimization is considered. For any t, $\mathcal{U}_{t,t+1} : \mathbb{R}^2_+ \mapsto \mathbb{R}$ is defined by

$$\mathcal{U}_{t,t+1}(k, m) \equiv -(\gamma^k_{t,t+1} - \beta\bar{z})k - (\gamma^m_{t,t+1} - \beta)\,\psi_{t+1}m + \beta\varsigma E_{G_{t+1}}(m, k, \psi_{t+1}),$$

where $\gamma^k_{t,t+1} \equiv p_t/(r_{t+1}p_{t+1})$ and $\gamma^m_{t,t+1} \equiv (\tau_t\phi_t)/\phi_{t+1}$. In addition, let $a_t \equiv \phi_t(\tau_t - 1) + \bar{h} + c(p_t)$ for all t. Through repeated substitution of (2), the following equation is obtained:

$$V_t(k, m) = \varsigma p_t r_t E_{G_{t+1}}(m, k, \psi_t) + p_t r_t \bar{z}k + \phi_t m + a_t$$

$$+\sum_{s=t}^{\infty}\beta^{t-s}[p_{s+1}r_{s+1}\max_{\mathcal{G}}\int\mathcal{U}_{s,s+1}(k',m')\mathcal{G}(\mathrm{d}(k',m'))+\beta a_{s+1}].$$

The equation shows that characterizing $\mathcal{G}_t$ is sufficient to investigate the two-period problem

$$\max_{k',m'}\mathcal{U}_{t,t+1}(k',m'). \tag{4}$$

Note that $\gamma^m_{t,t+1} = W_{t+1}/W_t$ indicates the gross inflation rate from day t to day $t+1$, measured by the nominal prices of leisure. In the absence of capital trade, the expected rates of return on capital and money in day t would be $\bar{z}/\gamma^k_{t,t+1} - 1$ and $1/\gamma^m_{t,t+1} - 1$. Evidently, these rates cannot exceed the time discount rate $1/\beta - 1$ in any equilibrium. At the inflation rate under the Friedman rule $\gamma^m_{t,t+1} = \beta$, holding money over night $t+1$ is costless. This is the only case in which a monetary equilibrium exists without using money at night.

Lemma 2. If (k', m') *is a solution to the problem (4) given* $\gamma^k_{t,t+1} \geq \beta\bar{z}$ *and* $\gamma^m_{t,t+1} \geq \beta$,

$$\psi_{t+1}m' \leq \min\{\mu^*, \hat{\mu}(\tilde{k}/k')\}k', \quad \forall\,\tilde{k} \in \mathrm{supp}(G_{t+1k}) \tag{5}$$

holding as an equality only if $\eta^* k' \leq \tilde{k}$ *for all* $\tilde{k}$ *in the support,* $\theta = 1$, *and* $\gamma^m_{t,t+1} = \beta$.

The preceding lemma implies that all bilateral trades occur under the Case II in equilibrium each night.[15] For notational brevity, $\hat{q}: \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ is defined by $\hat{q}(x, k) \equiv \hat{\eta}(x/k)k$, $\equiv 0$ if $k = 0$, and g: $\mathbb{R}^2_+ \mapsto \mathbb{R}_+$ by $g(x, k) \equiv f(\hat{q}(x, k), k)$. Here $g(0, k) = g(x, 0) = 0$ is trivial. From $g(x, k) \equiv \mathfrak{F}(\hat{\eta}(x/k))k$ for all $k \in \mathbb{R}_{++}$, g_1, $g_2 > 0$ is obtained. Unless Condition (5) holds as an equality, generality is not lost in letting

[15] This situation implies that the Friedman rule is still optimal. The liquidity constraint in purchasing capital binds unless $\theta = 1$ and $\gamma^m_{t,t+1} = \beta$; thus, relaxing the constraint yields an efficient allocation. See Figure 4 in the proof of the lemma. As a standard practice, a social optimum occurs when money holding has no cost due to inflation because it is costless to produce money.

$$E_G(m, k, \psi) \equiv zg(\psi m, k) - \psi m + \int [\psi\tilde{m} - \hat{q}(\psi\tilde{m}, \tilde{k})]\ G(\mathrm{d}(\tilde{k}, \tilde{m})).$$

In case that $\theta = 1$ and $\gamma^m_{t,t+1} = \beta$ for some t, we investigate the limit of equilibria as $\theta \to 1$ from below, or $\gamma^m_{t,t+1} \to \beta$ from above.[16] The Kuhn–Tucker necessary condition for a solution to Problem (4) is given by

$$\begin{aligned} \gamma^k_{t,t+1} - \beta\bar{z} &\geq \beta\varsigma z g_2(\psi_{t+1} m', k'), && = \text{if } k' > 0, \\ \gamma^m_{t,t+1} - \beta &\geq \beta\varsigma[z g_1(\psi_{t+1} m', k') - 1], && = \text{if } m' > 0. \end{aligned}$$

Given that $g(x, \cdot)$ satisfies the Inada conditions for all $x \in \mathbb{R}_{++}$, $k' > 0$ if k' is optimal because $\gamma^k_{t,t+1} > \beta\bar{z}$. However, $g_1(0, k) = \theta + (1 - \theta)/z$ for all $k \in \mathbb{R}_{++}$; thus, the unit cost of obtaining money in a day must be bounded from above for optimal $m' > 0$. A sufficient condition for a solution in $\mathbb{R}^2_{++}$ is as follows:

$$\gamma^k_{t,t+1} > \beta\bar{z}, \quad \beta \leq \gamma^m_{t,t+1} < \beta\varsigma\theta(z - 1). \tag{6}$$

The expected rates of return on capital and money without capital trade should not exceed the time discount rate. In addition, the unit cost of obtaining money in a day should not be too high to make it optimal to purchase no capital. Note that the existence of a solution in $\mathbb{R}^2_{++}$ does not guarantee $\gamma^m_{t,t+1} \geq \beta$. The equilibrium conditions derived from the Euler equations guarantee only the other inequalities.

For further analysis, this study focuses on the case that $\mathfrak{F} \circ \hat{\eta}$ is strictly concave. This concavity can be motivated by assuming that buyers of capital have a sufficiently strong bargaining power, that is, $\theta \approx 1$, and it implies that $g_{11} < 0$, $g_{22} < 0$, and $g_{12} > 0$. The Kuhn-Tucker condition for Problem (4) now completely characterizes its solution. Condition (6) is necessary and sufficient for an optimum in $\mathbb{R}^2_{++}$. Unlike the situation in Lagos, and Wright (2005), the concavity does not imply degenerate portfolio distributions. Given that g shows constant returns to scale, as in the standard profit maximization, optimality only pins down the ratio between k' and m'. However, an optimal ratio between the two is unique because of the concavity. Thus, the ratio must be

[16] Most reduced-form models do not have monetary equilibria under the Friedman rule. Thus, monetary studies commonly investigate the limit of equilibria as inflation approaches the Friedman rule instead. This study investigates equilibria under the Friedman rule as long as $\theta < 0$.

K_{t+1} to 1 for the money market clearing in day t. Therefore, the Euler equations for portfolio optimization yields as follows:

$$p_t = \beta r_{t+1} p_{t+1} [\varsigma z g_2(\psi_{t+1}, K_{t+1}) + \bar{z}], \tag{7}$$
$$\tau_t \phi_t = \beta \phi_{t+1} [\varsigma z g_1(\psi_{t+1}, K_{t+1}) + 1 - \varsigma]. \tag{8}$$

In addition, the goods market clearing condition (3) yields as follows:

$$c(p_t) + K_{t+1} = [\bar{z} K_t + \varsigma T(\psi_t, K_t)][F(1, \hbar(p_t)) + 1 - \delta], \tag{9}$$

where $T: \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is defined by $T(x, k) \equiv zg(x, k) - \hat{q}(x, k)$. The following lemma implies the law of motions for aggregate capital and daytime prices in an equilibrium can be described by a first-order difference equation system. Let $\boldsymbol{s}_t \equiv (K_t, p_t, \phi_t)$ for all t.

Lemma 3. *A continuously differentiable function $\Phi: \mathbb{R}_{++}^4 \mapsto \mathbb{R}_{++}^3$ exists, such that $\boldsymbol{s}_{t+1} = \Phi(\boldsymbol{s}_t, \tau_t)$ iff $(\boldsymbol{s}_{t+1}, \boldsymbol{s}_t, \tau_t) \in \mathbb{R}_+ \times \mathbb{R}_{++}^6$ satisfies (7), (8), and (9) simultaneously.*

Given G_0 and $\{\tau_t\}_{t=0}^{\infty}$, an equilibrium is completely characterized by a bounded sequence $\{\boldsymbol{s}_t\}_{t=0}^{\infty}$ satisfying the first-order difference equation given by Φ and $\beta\phi_{t+1} \le \tau_t \phi_t$ for all $t \in Z_+$ under the initial condition $\boldsymbol{s}_0 \in \mathbb{R}_{++}^3$. The portfolio distribution G_t for any t is arbitrary as long as it yields the aggregate capital K_t in the sequence. If $\{\tau_t\}_{t=0}^{\infty}$ follows a recursive process, the standard recursive equilibrium can be defined for the model.

IV. Steady-State Inflation

This section analyzes the effects of anticipated inflation on a unique steady-state equilibrium.

A steady-state given gross rate of money growth $\tau \ge \beta$ is $\boldsymbol{s} \equiv (K, p, \phi) \in \mathbb{R}_{++}^3$ such that $\boldsymbol{s} = \Phi(\boldsymbol{s}, \tau)$. Given $\boldsymbol{s}_t = \boldsymbol{s}$, $\tau_t = \tau$ for all t, the gross inflation rate P_{t+1}/P_t, as well as $\gamma_{t,t+1}^m$, is constant at τ, and $\gamma_{t,t+1}^k$ is constant at $r \equiv F_1(1/\hbar(p), 1) + 1 - \delta$. For simplicity in analysis, c is redefined as a function of $\hbar = \hbar(p)$, implied by $u'(c(\hbar)) F_2(1, \hbar) = 1$. Moreover, $\mu \equiv (rp)^{-1} \phi / K$ denotes the fraction of aggregate capital that is traded in the capital market. A steady state is completely characterized by $(K, \hbar, \mu) \in \mathbb{R}_{++}^3$ simultaneously satisfying

$$1 = \beta[\varsigma z g_2(\mu, 1) + \bar{z}][F_1(1/\hbar, 1) + 1 - \delta], \tag{10}$$
$$\tau = \beta[\varsigma z g_1(\mu, 1) + 1 - \varsigma], \tag{11}$$
$$c(\hbar) = [\zeta(\mu, \hbar) - 1]K, \tag{12}$$

where $\zeta\colon \mathbb{R}_+^2 \mapsto \mathbb{R}_{++}$ is defined by

$$\zeta(\mu, \hbar) \equiv [\bar{z} + \varsigma T(\mu, 1)][F(1, \hbar) + 1 - \delta]$$

to represent the output (including the scrap value) per physical unit of capital.

Proposition 1. *Given $\tau \geq \beta$, a unique steady state exists iff $\tau < \beta\varsigma\theta(z - 1)$.*

Condition (11) provides that an increase in the money growth rate or the inflation rate τ reduces the proportion of capital traded μ becuase $g_{11} < 0$. Given the rule for the terms of capital trade, installation technology implies the diminishing rate of technical substitution between money and capital in production for additional capital input. Hence, an increase in the cost of holding money induces the portfolio substitution out of money into capital. This condition represents the Tobin effect.[17] Note that μ is not the real money balance. A reduction in μ does not imply the same size of a reduction in the quantity of capital traded as a reduction in the real money balance does in a reduced-form model.

Condition (10) provides that a reduction in μ increases the employment per effective unit of capital $\hbar$ because $g_{12} > 0$, $F_{11} < 0$. In addition, consumption c decreases as $\hbar$ increases. The substitution out of money into capital increases the investment demand, and households reduce consumption and increase work to save additional capital.[18] The

[17] This effect and the other succeeding results do not depend on the assumption of constant returns to scale in the installation as long as the production for additional capital input exhibits diminishing marginal products of money and capital, and money and capital are technical complements in this production.

[18] In a standard cash-in-advance model, such as that of Cooley, and Hansen (1989), consumption is a cash good, whereas leisure is a credit good. Inflation increases the cost of holding money. Thus, in such model, it reduces consumption and increases leisure (thereby decreasing labor supply). However, in the model of the current study, it induces households to switch investment from money holding to capital accumulation. Thus it reduces leisure and

increases in labor supply and demand for the general good indicate that the real wage per hour $1/p$ decreases and the real profit per effective unit of capital r increases. Note that the strength of this *price distorting effect* increases with the technical complementarity in production for additional capital input, that is, g_{12}. A high complementarity or low substitutability implies increased distortion.

The change in the output per physical unit of capital ζ depends on the direct negative effect of the reduction in μ and the positive price distorting effect of this reduction. Around the steady state,

$$\frac{d\zeta}{d\tau} = \hat{\eta}'(\mu)[z\mathfrak{F}'(\hat{\eta}(\mu)) - 1][F(1, \hbar) + 1 - \delta] \frac{d\mu}{d\tau} + [\bar{z} + \varsigma T(\mu, 1)]F_2(1, \hbar)\frac{d\hbar}{d\tau} \quad (13)$$

with $d\mu/d\tau < 0$ and $d\hbar/d\tau = (d\hbar/d\mu)(d\mu/d\tau) > 0$, and $d\hbar/d\mu$, $d\mu/d\tau$ can be easily obtained by applying the implicit function theorem to Conditions (10) and (11). If the price distorting effect is stronger than the direct one, then the output per capital increases. However, as will be shown below, such situation happens when $\hbar$ increases mainly because of the reduction in effective capital rather than the increase in labor employment.

Given the change in ζ and the reduction in c, the change in the aggregate capital stock K is completely characterized by Condition (12). The change in output $Y = \zeta(\mu, \hbar)K$ is equal to the sum of the changes in K and c. Around the steady state,

$$[\zeta(\mu, \hbar) - 1] \frac{dK}{d\tau} = c'(\hbar) \frac{d\hbar}{d\tau} - K\frac{d\zeta}{d\tau}.$$

If the output per capital increases, then the aggregate capital and output are reduced because consumption decreases (see the left panel of Figure 2). The increase in the output per capital indicates low substitutability between money and capital in production for additional capital input. If money and capital are sufficiently substitutable in the production, and thus, the direct effect of the reduction in μ is sufficiently strong, then a rise in the inflation rate increases the aggregate capital and output (see the right panel of Figure 2). To be

consumption because production and savings must increase to meet the increased demand for investment goods.

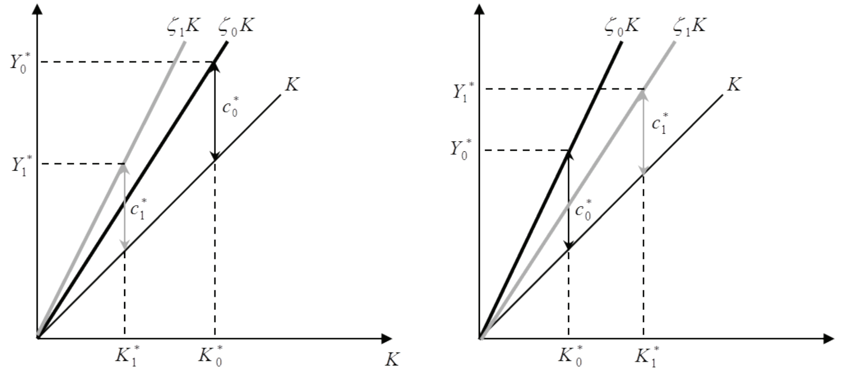

FIGURE 2
THE EFFECTS OF INFLATION

concrete, suppose that $g_{12} \approx 0$. Accordingly, the price distorting effect is negligible, that is, $d\hbar/d\tau \approx 0$, and the consumption decreases slightly. Condition (12) implies that the aggregate capital increases nearly as much as to compensate the reduction in the output per capital by the direct effect of the reduction in μ. The output also increases nearly as much as the aggregate capital increases.

Although inflation may increase the output, it never improves the welfare of households. Households work too much and consume too small at any inflation rate other than the Friedman rule $\tau = \beta$, and an increase in the rate enhances distortion. Unlike existing studies that only take negative wealth effect into account, the current study demonstrates the possibility that households work excessively under inflation.

Investigate how the degree of search frictions (or matching technology) in the capital market affects the effects of inflation. Condition (10) indicates that μ becomes more responsive to changes in τ as the trading probability ς decreases. If search frictions are large, then a heavy inflation tax burden is imposed on each trade in it, thereby strengthening the direct substitution effect. By contrast, Condition (11) implies that the price distorting effect is withered if search frictions are large. Both $\hbar$ and c become less responsive to changes in μ as ς falls. Moreover, Condition (13) shows that ζ becomes less responsive to changes in $\hbar$ as ς falls. Therefore, an increase in the inflation rate will likely increase the aggregate capital and output when search frictions are large.

V. Conclusion

A model with a decentralized capital market is developed to analyze the effects of monetary policy on the production side. This model provides a search-theoretic account for the link between liquidity problems in firms' production and the supply of monetary liquidity.

In installing additional capital for immediate use, money and capital function as if they are two factors of capital production. By increasing the cost of holding money, inflation exerts a direct positive effect on the portfolio substitution out of money into capital and a negative price distorting effect on the aggregate capital and output. If money and capital are strong substitutes in the installation, then the positive direct effect dominates the negative one. This case will likely occur when search frictions in the capital market are large. This study focuses on the market structure used in most of the recent literature on the microfoundations of money. However, the key messages of this research do not seem to change as long as the structure yields the diminishing rate of technical substitution between money and capital in the installation.

Future studies may focus on several aspects. First the proposed model can be calibrated with specific functional forms for quantitative findings, such as the numerical effects of monetary policy on capital formation, welfare, and other macro variables. Then, as usual in new monetarist studies, one can examine how different pricing mechanisms influence the effects of monetary policy. In numerical analysis, introducing aggregate productivity shocks and investigating the fluctuations around the steady state are not difficult. In aspect of theoretical development, the proposed model can be extended to incorporate investment banking. Then one can take more seriously the details of central banking with a loan market in which investment banks compete.

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Appendix

A. Proof of Lemma 1

Take arbitrary $k_b \in \mathbb{R}_{++}$, and fix it. Thereafter, consider the maximization problem

$$\max_{\eta,\,\mu}[z\mathfrak{F}(\eta)-\mu]^{\theta}[\mu-\eta]^{1-\theta} \tag{14}$$

subject to $0 \leq \eta \leq \eta_0 \equiv k_s/k_b$ and $0 \leq \mu \leq \mu_0 \equiv \psi_t\, m_b/k_b$. A pair of $q=\eta k_b$ and $\ell_t = \mu/k_b$ is a solution to (1) if, and only if, (η, μ) is a solution to (14). The first order condition (FOC) for (14) is as follows:

$$\mu-\hat{\mu}(\eta)=\lambda_{\eta}[z\mathfrak{F}(\eta)-\mu]^{\theta}[\mu-\eta]^{1-\theta}[\theta z\mathfrak{F}'(\eta)+1-\theta],$$
$$\breve{\mu}(\eta)-\mu=\lambda_{\mu}[z\mathfrak{F}(\eta)-\mu]^{\theta}[\mu-\eta]^{1-\theta},$$

where λ_η and λ_μ are the Lagrange multipliers on $\eta \leq \eta_0$ and $\mu \leq \mu_0$, respectively.

In addition to the properties of $\hat{\mu}$, $\breve{\mu}$ listed in the text, we have

$$\breve{\mu}(\eta)-\hat{\mu}(\eta)=\theta(1-\theta)[z\mathfrak{F}(\eta)-\eta][z\mathfrak{F}'(\eta)-1][\theta z\mathfrak{F}'(\eta)+1-\theta]^{-1};$$

thus, $\breve{\mu}(\eta) \geq \hat{\mu}(\eta)$ for every $\eta \in (0, \eta^*)$, with equality only when $\theta = 1$. FOC implies that $\eta \geq \hat{\eta}(\mu)$ because $\lambda_\eta \geq 0$, and that $\mu \geq \breve{\mu}(\eta)$ because $\lambda_\mu \geq 0$. Hence, we have four cases in the lemma (see the left panel of Figure 3), and the lemma directly follows from the complementary slackness conditions. For example, in Case II, we have $\mu = \mu_0$ and $\eta = \hat{\eta}(\mu_0)$ from $\lambda_\mu > 0$ and $\lambda_\eta = 0$ (see the right panel of Figure 3). ■

B. Proof of Lemma 2

We focus on an optimal choice of money m' in day t, given a choice of capital $k' = k_0 > 0$, as well as ψ_{t+1}, $\gamma^k_{t,t+1}$, $\gamma^m_{t,t+1}$, and G_{t+1}. For technical simplicity, we investigate a choice of $x \equiv \psi_{t+1} m'$ instead. In addition, we assume that G_{t+1k} is discrete with the support $k_1 < k_2 < \cdots < k_I$, and let $\eta_i \equiv k_i/k_0$ denote the ratio of k_i to k_0.

First we consider the case that $\eta_1 \geq \eta^*$, that is, $\mu^* \leq \hat{\mu}(k_i/k_0)$ for all i. Any $x_0 > \mu^* k_0$ is not optimal unless $\gamma^m_{t,t+1} = \beta$. A reduction in x from x_0 by sufficiently small Δx raises the expected utility by $(\gamma^m_{t,t+1} - \beta)\Delta x$. In case that $\gamma^m_{t,t+1} = \beta$, any $x_0 > \mu^* k_0$ is indifferent from $x = \mu^* k_0$. However a reduction in x from $\mu^* k_0$ by small Δx raises the expected utility by

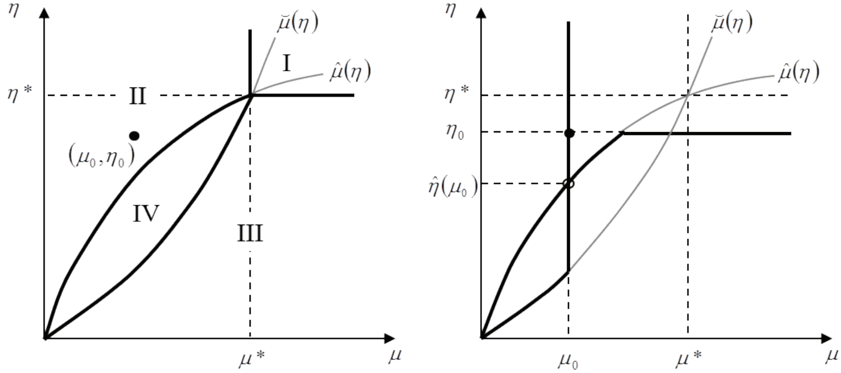

FIGURE 3
BARGAINING SOLUTION

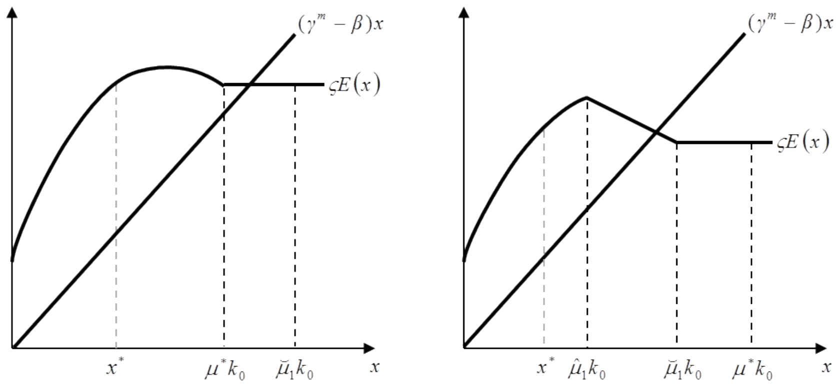

FIGURE 4
MONEY DEMAND

$$(\gamma^m_{t,t+1} - \beta)\Delta x + \beta\varsigma[1 - [1 - \theta(1 - \theta)z\mathfrak{F}''(\eta^*)[z\mathfrak{F}(\eta^*) - \eta^*]]^{-1}]\Delta x.$$

The second term equals 0 if $\theta = 1$; otherwise, it is strictly positive. Thus, if (x^*, k_0) is optimal, then $x^* \leq \mu^* k_0$ with equality only if $\theta = 1$, $\gamma^m_{t,t+1} = \beta$ (see the left panel of Figure 4).

Next we consider the case that all households choose the same amount of capital k_1 ex post, that is, G_{t+1k} degenerates to k_1, and $\eta_1 < \eta^*$. If a certain household chose $x_0 > \hat{\mu}(\eta_1)k_0$, then a reduction in x from x_0 by small Δx would raise the expected utility by $(\gamma^m_{t,t+1} - \beta)\Delta x$ if $x_0 > \breve{\mu}(\eta_1)$

k_0; otherwise, $(\gamma^m_{t,t+1} - \beta)\Delta x + \beta\varsigma\Delta x$. Hence, such x_0 is not optimal (see the right panel of Figure 4). We define R: $\mathbb{R}_+ \mapsto \mathbb{R}$ by

$$R(\mu) \equiv -(\gamma^k_{t,t+1} - \beta\bar{z})[1 + \rho_{t,t+1}\,\mu] + \beta\varsigma[z\mathfrak{F}(\hat{\eta}(\mu)) - \mu]$$

with $\rho_{t,t+1} \equiv (\gamma^m_{t,t+1} - \beta)\,(\gamma^k_{t,t+1} - \beta\bar{z})^{-1}$. If a certain household chose $x = \hat{\mu}(\eta_1)\,k_0$, then a reduction in k' from k_0 by small $\Delta k'$ with the same percentage reduction in x, which maintains the ratio of x to k' constant at $\hat{\mu}(\eta_1)$, would raise the expected utility by $-R(\hat{\mu}(\eta_1))\Delta k'$. Thus, $(\hat{\mu}(\eta_1)\,k_0,\ k_0)$ can be optimal only if $R(\hat{\mu}(\eta_1)) > 0$. However, an increase in k' from k_0 by small $\Delta k'$ with the same percentage increase in x would raise the expected utility by

$$R(\hat{\mu}(\eta_1))\Delta k' + \beta\varsigma[z\eta_1\mathfrak{F}'(\eta_1) + 1]\Delta k' > 0.$$

Thus, $(\hat{\mu}(\eta_1)\,k_0,\ k_0)$ cannot be optimal. If $(x^*,\ k^*)$ is optimal, then $x^* < \hat{\mu}(\eta_1)\ k^*$.

The remaining is to prove the lemma for the case $I \geq 2$ with $\eta_1 < \eta^*$. We consider the case that $I = 2$. Let x^* be an optimal choice of x when all the households choose the same amount of capital k_2. Then $x^* < \min\{\mu^*,\ \hat{\mu}(1)\}\,k_2$. In addition, the optimality of $(x^*,\ k_2)$ implies $R(\mu^*) = 0$ with $\mu^* \equiv x^*/k_2$. Suppose that a household chooses $(x_0,\ k_0)$ such that $x_0 \geq \hat{\mu}(\eta_1)\,k_0$, and that x_0 is optimal given k_0. In this case, a lack of capital of its own hinders a further increase in the expected utility from using money. Increasing k' from k_0 to k_2 raises the expected utility in the choice of x. In addition, no loss from the increase in capital is incurred if the ratio of x to k' is changed to μ^*. Hence, if $(x^*,\ k^*)$ is optimal, then $x^* < \hat{\mu}(\eta_1)\,k^*$. The mathematical induction completes the proof. ■

C. Proof of Lemma 3

Let $\boldsymbol{s}^*_t$ denote $(\boldsymbol{s}_{t+1},\ \boldsymbol{s}_t,\ \tau_t) \in \mathbb{R}_+ \times \mathbb{R}^6_{++}$ satisfying Equations (7), (8), and (9) together.

The market clearing condition Equation (9) directly gives the law of motion for aggregate capital $\Phi_k : \mathbb{R}^3_{++} \mapsto \mathbb{R}_+$ such that $K_{t+1} = \Phi_k(\boldsymbol{s}_t)$ for all $\boldsymbol{s}^*_t$. In addition, given any $p_t \in \mathbb{R}_{++}$, no $p_{t+1} < \infty$ satisfies Equation (7) if $K_{t+1} = 0$. Hence, the codomain of Φ_k can be restricted to $\mathbb{R}_{++}$.

Let $\mathbf{p}_t \equiv (p_t,\ \phi_t)$ denote the price vector in day t, and focus on finding $\Phi^{\mathbf{p}} : \mathbb{R}^4_{++} \mapsto \mathbb{R}^2_{++}$ such that $\mathbf{p}_{t+1} = \Phi^{\mathbf{p}}(\mathbf{p}_t,\ \tau_t,\ K_{t+1})$ for all $\boldsymbol{s}^*_t$. First, we define $\gamma^k,\ \gamma^m : \mathbb{R}^2_{++} \mapsto \mathbb{R}_{++}$ by

$$\gamma^k(x, k) \equiv \beta[\varsigma z g_2(x, k) + \bar{z}], \quad \gamma^m(x, k) \equiv \beta[\varsigma z g_1(x, k) + 1 - \varsigma].$$

Equations (7) and (8) imply $\gamma^k_{t,t+1} = \gamma^k(\psi_{t+1}, K_{t+1})$ and $\gamma^m_{t,t+1} = \gamma^m(\psi_{t+1}, K_{t+1})$; hence, by the definitions of $\gamma^k_{t,t+1}$, $\gamma^m_{t,t+1}$ and ψ_{t+1}, we have

$$\gamma^m(\psi_{t+1}, K_{t+1}) = \psi_{t+1}\gamma^k(\psi_{t+1}, K_{t+1})\, \tau_t\phi_t/p_t, \quad \forall\, \mathbf{s}_t^*.$$

From $g_{12} > 0$, $g_{11} < 0$, we obtain $\gamma^k_1 > 0$, $\gamma^m_1 < 0$. In addition, $\gamma^k(0, k) = \infty$, $\gamma^k(\infty, k) = \beta\bar{z}$, $\gamma^m(0, k) = \beta\varsigma\theta(z-1)$, and $\gamma^m(\eta^* k, k) \leq 0$ for all $k \in \mathbb{R}^2_{++}$. Thus, given any $(\mathbf{p}_t, \tau_t, K_{t+1}) \in \mathbb{R}^4_{++}$, there exists unique $\psi_{t+1} \in \mathbb{R}_{++}$ satisfying the preceding equation; hence a unique $\phi_{t+1} \in \mathbb{R}_{++}$ exists. In addition, since r_{t+1} is strictly increasing in p_{t+1}, there exists unique $p_{t+1} \in \mathbb{R}^2_{++}$ such that $\gamma^k_{t,t+1} = \gamma^k(\psi_{t+1}, K_{t+1})$. Therefore, $\Phi^{\mathbf{p}}$ is well defined, and the law of motion for price vector $\Phi_{\mathbf{p}} : \mathbb{R}^4_{++} \mapsto \mathbb{R}^2_{++}$ is given by $\Phi_{\mathbf{p}}(\mathbf{s}_t, \tau_t) \equiv \Phi^{\mathbf{p}}(\mathbf{p}_t, \tau_t, \Phi_k(\mathbf{s}_t))$.

The laws of motion $\Phi_{\mathbf{p}}$ and Φ_k constitute Φ in the lemma. The continuous differentiability derives from the implicit function theorem. ■

D. Proof of Proposition 1

Evidently, given $\tau \geq \beta$, a unique $\mu \in \mathbb{R}_{++}$ satisfying Equation (11) exists if $\tau < \beta\varsigma\theta(z-1)$. If $\mu \in \mathbb{R}_{++}$ exists, then a unique $\hbar \in \mathbb{R}_{++}$ satisfying Equation (10) exists.

The proof is completed by showing that $\zeta(\mu, \hbar) > 1$. Given that g shows constant returns to scale,

$$T(\mu, 1) = zg(\mu, 1) - \hat{\eta}(\mu) = zg_2(\mu, 1) + \mu[zg_1(\mu, 1) - 1] + \mu - \hat{\eta}(\mu).$$

From Equation (11), we have $zg_1(\mu, 1) > 1$. In addition, $\mu \geq \hat{\eta}(\mu)$ because $(\hat{\eta}(\mu), \mu)$ is a solution to Problem (14). Thus, $T(\mu, 1) > zg_2(\mu, 1)$. Given that F shows constant returns to scale, $F(1, \hbar) > F_1(1/\hbar, 1)$, and $\zeta(\mu, \hbar)$ is larger than the right hand side of Equation (10). ■

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