A Stability Analysis of a Multiple Denomination Equilibrium

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We consider existence and stability in a simple version of Lee *et al.* (2005). For a sufficiently large discount factor, a steady state of full – support wealth distribution and pure strategy exists, whereas for an intermediate discount factor, a steady state of mixed strategy exists. Both steady states are locally stable and determinate. All denominations are circulated in the mixed-strategy steady state, whereas larger denominations are not held in the other steady state. We also show that nonfull-support steady states exist, and are stable and indeterminate. This finding is in sharp contrast to that of Huang, and Igarashi (2014), which show instability of nonfull-support steady states in Lee *et al.* (2005)

Keywords: Random matching model, Carrying cost, Denomination, Instability, Determinacy

JEL Classification: C62, C78, E40

I. Introduction

Lee *et al.* (2005) are the first to model the denomination of currency and the trade-off between carrying cost and flexible transaction opportunities that a small denomination permits. A suitable framework to model that trade-off is a model in which money is essential and indivisible. Therefore, Lee *et al.* (2005) therefore build their model based on Zhu, who, in turn, extended Trejos, and Wright (1995) to a larger wealth set. Each date has a portfolio choice stage in which individuals can

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[**Seoul Journal of Economics** 2015, Vol. 28, No. 4]

choose what form they want to hold their wealth (*e.g.*, one \$5 *vs.* five \$1's), which is followed by a standard matching and trading stage. Lee *et al.* (2005) rule out several methods to avoid carrying costs, such as the electronic payment system in Huang *et al.* (2015). In such an environment they show existence of steady states. We show more detailed existence and stability results in a simple version of their framework.

Lee *et al.* (2005) allow for an arbitrary bound on wealth. We provide an analysis of stability in the simplest case, where the bound is 2. In this economy, people's wealth is in set $\{0, 1, 2\}$, and there are only two denominations, namely, \$1 and \$2 bills. Despite its simplicity, this setup has all the essence of Lee *et al.* (2005). In particular, when an individual who has wealth 2 chooses his portfolio, he encounters a trade-off or dilemma of denomination. That is, while choosing one \$2 bill incurs less carrying cost than choosing two \$1 bills, this choice may lead to a loss of trading opportunity; if he meets a seller without a \$1 bill, he will not have the option to spend only \$1 because the seller cannot offer change.

We study the full-support steady states, in which wealth distribution has support $\{0, 1, 2\}$. Full-support steady state has two types, one supported by a pure strategy and the other by a mixed strategy. In the pure-strategy steady state, all the wealth is held in the form of \$1 bills and no one holds a \$2 bill, which serves as an example in which the largest unit (*e.g.*, \$100 bill in the US) does not circulate. In the mixed-strategy steady state, both types of bills circulate. Considering that Lee *et al.* (2005) do not respond whether the smallest units and larger units can circulate at the same time in equilibrium, the above mixed-strategy steady state provides the first example in which a larger unit essentially matters. Finally, both types of full-support steady states are shown to be stable and determinate. To the best of our knowledge, our approach to the stability of the mixed-strategy steady state is different from existing literature.

We also consider two types of non-full-support steady states, in which wealth distribution has support $\{0, 2\}$. One type involves converting \$1s into \$2s, whereas the other involves discarding \$1s. Both types are stable. The economy, deviating slightly from one type, can jump back to it by discarding \$1 bills, whereas the economy, deviating from the other type, can gradually converge to the other by means of converting all \$1s into \$2s. This effect is in sharp contrast to the case of zero carrying cost, where the dilemma of denomination does not occur. Specifically, Huang, and Igarashi (2014) show that such gradual convergence is impossible for the non-full-support steady state in Zhu.

II. Model

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Let $s=(s_1, s_2, \ldots, s_K)$ be a denomination structure of money, where s_k is the size of the k-th smallest denomination. Assume $s_{k-1}<s_k$ and that s_k/s_1 is an integer. We normalize s_1 to be 1 and impose a bound on individual nominal wealth Z for technical purposes. Denote the set of nominal wealth $\mathbb{Z}=\{1, \ldots, Z\}$. Let $m\in(0, 1)$ be the per capita wealth divided by the bound Z. The time discount factor is $\beta\in(0, 1)$, and the period utility is $(c)-q-\gamma\sum_k y_k$, where $c\in\mathbb{R}_+$ is the amount of consumption, $q\in\mathbb{R}_+$ is the amount of production, and $\gamma\geq 0$ is the utility cost of carrying money of any size from the portfolio choice stage to the pairwise stage. Function $u\colon\mathbb{R}_+\rightarrow\mathbb{R}_+$ has all nice properties: $u(0)=0$, $u'(\infty)=0$, $u'>0$, $u''<0$, and sufficiently large but finite $u'(0)$.

In the portfolio choice stage, agents can choose any portfolio of monetary items within their wealth. Define the set of feasible portfolios, $\mathbb{Y}=\{y=(y_1, \ldots, y_k)\in\mathbb{R}_+^K\colon sy\leq Z\}$, where y_k is the integer quantity of size-s_k money. Let $\Gamma_1(z)$ be a subset of probability measures on the feasible portfolios of nominal wealth less than z:

$$\Gamma_1(z)=\{\sigma_1\colon \sigma_1(y)=0, \quad \text{if} \quad sy>z\}. \tag{1}$$

Let $h^t\colon\mathbb{Y}\rightarrow\mathbb{R}$ be the value function that gives the expected discounted utility of entering pairwise meetings at date-t with portfolio y, and let $w^t\colon\{0, 1, \ldots, Z\}\rightarrow\mathbb{R}$ be the utility of entering date t with wealth z. Hence, the optimization problem at the portfolio stage is

$$w^t(z)=\max_{\sigma_1\in\Gamma_1(z)}\sum\sigma_1(y)h^t(y) \tag{2}$$

Denote the set of maximizers in Equation (2) by $\Delta_1^t(z)$. Equation (2) implies that agents can choose any portfolio in $\Gamma_1(z)$ at no cost. And we then allow $sy<z$ in Equation (1). Hence free discarding of money is allowed at this stage. In the pairwise meeting, agents become a buyer (the partner becomes a seller) with probability $1/n$. With probability $1-$

$(2/n)$, the match is a no-coincidence meeting. In single-coincidence meetings, the buyer makes a take-it-or-leave-it offer. This offer consists of production amount, the monetary items the buyer should transfer, and the monetary items the seller should transfer ("change"). Randomization is allowed. However, with risk-averse agents, the optimal production amount is degenerated. Define the set of feasible wealth transfers from the buyer with y to the seller with y':

$$X(y, y') = \{x \in [0, Z - sy']: x = s(v - v'),\ 0 \leq v \leq y,\ 0 \leq v' \leq y'\}, \tag{3}$$

where the inequalities are about vector comparison. Given any transfer of monetary wealth $x \in X(y, y')$, the optimal production is

$$q^t(x) = \beta w^{t+1}(x + sy') - \beta w^{t+1}(sy'). \tag{4}$$

Let $\Gamma_2(y, y')$ be the set of all probability measures on $X(y, y')$. The problem in pairwise trade between a buyer with y and a seller with y' at date t is

$$f^t(y, y') = \max_{\sigma_2 \in \Gamma_2(y, y')} E_{\sigma_2}[u(q^t(x)) + \beta w^{t+1}(sy - x)], \tag{5}$$

where randomization over monetary payments is allowed. Denote the set of maximizers in Equation (5) by $\Delta_2^t(y, y')$.

Let π^t be the probability measure on $\mathbb{Z}$, such that $\pi^t(z)$ is the fraction of each type with wealth z at the start of date t. Let θ^t be the probability measure on $\mathbb{Z}$, such that $\theta^t(y)$ is the fraction of each type with portfolio y right after the portfolio stage at date t. These distributions and value functions must satisfy all the laws of motions and the Bellman Equations in Lee *et al.* (2005):

$$h^t(y) = -\gamma \textstyle\sum_k y_k + \dfrac{N-1}{N} \beta\, w^{t+1}(sy) + \dfrac{1}{N} \textstyle\sum_{y' \in \mathrm{Y}} \theta^t(y')\, f^t(y, y') \tag{6}$$

$$\begin{aligned} \pi^{t+1} \in \{\pi: \pi(z) = \frac{1}{N}\ & \textstyle\sum_{(y, y')}^{\square} \theta^t(y)\theta^t(y')[\sigma_{(y, y')}(sy - z) + \sigma_{(y', y)}(z - sy)] \\ + \frac{N-2}{N} \textstyle\sum_{sy=z} \theta^t(y),\ & \text{with } \sigma_{(y, y')} \in \Delta_2^t(y, y'),\ \text{for all } (y, y') \in \mathrm{Y} \times \mathrm{Y}\} \end{aligned} \tag{7}$$

$$\theta^t \in \{\theta: \theta(y) = \textstyle\sum_z \pi^t(z)\sigma_z(y),\ \text{with } \sigma_z \in \Delta_1^t(z),\ \text{for all } z \in \mathbb{Z}\}. \tag{8}$$

When a non-stationary, dynamic equilibrium is considered, the initial condition π^0, which is the distribution of wealth prior to the portfolio stage at $t=0$, is given.

Definition 1: Given π^0, an equilibrium is a sequence $\{(\theta^t, \pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies conditions (1)-(8). A tuple (θ, π, w) is a monetary steady state if $(\theta^t, \pi^t, w^t)=(\theta, \pi, w)$ for $t \geq 0$ is an equilibrium and $w \neq 0$. Pure-strategy steady states are ones where both $\Delta_1(z)$ and $\Delta_2(y, y')$ are singletons[1] for all $z \in \mathbb{Z}$ and $(y, y') \in \mathrm{Y} \times \mathrm{Y}$. Other steady states are called mixed-strategy steady states.

In other words, pure-strategy steady states have a unique optimal choice in the portfolio stage and in all pairwise meetings. Mixed-strategy steady states can have degenerate mixed strategies, but they must have multiple optimal choices in both stages. Next, we define stability.

Definition 2: A steady state (θ, π, w) is locally stable if there is a neighborhood of π such that for any initial distribution in the neighborhood, there is an equilibrium path such that $(\theta^t, \pi^t, w^t) \rightarrow (\theta, \pi, w)$. A locally stable steady state is determinate if for each initial distribution in this neighborhood, there is only one equilibrium that converges to the steady state.

III. Full-support steady states

The economy we analyze has wealth bound $Z=2$; hence the per capita wealth is $2m$. Hereafter, we use superscripts for date and subscripts for state on portfolio holding or wealth holding. Consequently, wealth distribution is $\pi=(\pi_0, \pi_1, \pi_2)$, and the value function of holding monetary wealth at the beginning of the portfolio choice stage is summarized in $\underline{w}=(w_0, w_1, w_2)$. The denomination structure of the economy is {\$1 bill, \$2 bill}. The beginning of the pairwise stage has four possible individual states: $\mathrm{Y} \equiv \{0, \$1, 2\$1\mathrm{s}, \$2\}$, where \$1 and \$2 represent the holding of one \$1 bill and one \$2 bill respectively; and 2\$1s represents the holding of two \$1 bills. Consequently, the distribution and value function over these four states are $\theta=(\theta_0, \theta_1, \theta_{11}, \theta_2)$ and $h=(h_0, h_1, h_{11}, h_2)$ respectively, where the subscripts 1, 11, and 2 indicate \$1, 2\$1s, and \$2, respect-

[1] If both $\Delta_1(z)$ and $\Delta_2(y, y')$ are singletons, both the optimal trades in Equation (1) and (5) are degenerated.

ively. The normalization $u(0)=0$ and the buyer take-it-or-leave-it offer imply $w_0=h_0=0$.

A (b, s)-meeting is a meeting between a buyer with state $b\in\Upsilon$ and a seller with state $s\in\Upsilon$. Trade and positive production may occur in five types of meetings: (\$1, 0)-meeting, (\$1, \$1)-meeting, (\$2, \$1)-meeting, (\$2, 0)-meeting, and (2\$1s, 0)-meeting. Notice that in the (\$2, \$1)-meeting, the buyer can transfer exactly \$1 by the seller offering change. (2\$1s, \$1)-meeting and (\$2, \$1)-meeting share the same set of possible trading opportunities in terms of transferring nominal wealth. (2\$1s, 0)-meeting has more possible trading opportunities than (\$2, 0)-meeting. In other meetings, buyers with two \$1 bills have the same trading opportunities as those with one \$2 bill.

For all the steady states of our interest, $\theta_0=\pi_0$, $\theta_1=\pi_1$ and $\theta_{11}+\theta_2=\pi_2$. The Bellman equations are as follows

$$h_1=\frac{n-1+\pi_2}{n}\beta w_1+\frac{\pi_0}{n}\max\{u(\beta w_1),\beta w_1\}+\frac{\pi_1}{n}\max\{u(\beta w_2-\beta w_1),\beta w_1\}-\gamma$$

$$\begin{aligned}h_{11}=&\frac{n-1+\pi_2}{n}\beta w_2+\frac{\pi_0}{n}\max\{u(\beta w_2),u(\beta w_1)+\beta w_1,\beta w_2\}\\&+\frac{\pi_1}{n}\max\{u(\beta w_2-\beta w_1)+\beta w_1,\beta w_2\}-2\gamma\end{aligned}\tag{9}$$

$$\begin{aligned}h_2=&\frac{n-1+\pi_2}{n}\beta w_2+\frac{\pi_0}{n}\max\{u(\beta w_2),\beta w_2\}+\frac{\pi_1}{n}\max\{u(\beta w_2-\beta w_1)\\&+\beta w_1,\beta w_2\}-2\gamma,\end{aligned}$$

and

$$\begin{aligned}w_1&=\max\{h_1,0\}\\w_2&=\max\{h_{11},h_2,0\},\end{aligned}\tag{10}$$

where the max operators correspond to decision problems in the portfolio and pairwise stages.

Disposing \$1s does not happen in some steady states, namely, steady states with full-support wealth distribution. Two such steady states include one with a pure strategy and one with a mixed strategy. Tables 1 and 2 show the equilibrium strategies, the bill chosen by the agents at the portfolio choice stage, and the offers of the buyer to the seller at the pairwise stage. In the mixed strategy, randomization occurs in the port-

TABLE 1
PURE STRATEGY

Portfolio stage	Always choose \$1 bills				
Pairwise stage			Seller's wealth		
			0	1	2
	Buyer's wealth	0	\	\	\
		1	\$1	\$1	\
		2	\$1	\$1	\

TABLE 2
MIXED STRATEGY

Portfolis stage	Randomize over 2\$1s and 1\$2 if Wealth is 2; choose \$1 otherwise					
Pairwise stage				Seller's wealth		
				0	1	2
	Buyer's wealth	0	\	\	\	\
		1	\	\$1	\$1	\
		2	2\$1s	\$1	\$1	\
			1\$2	\$2	\$1	\

folio choice stage (*i.e.*, $h_{11}=h_2$) but not in the pairwise stage. Particularly, agents with wealth 2 randomize over two \$1 bills and one \$2 bill in the portfolio stage. Those who choose to hold one \$2 bill offer \$2 in a (\$2, 0)-meeting, and those who choose to hold two \$1 bills offer only \$1 in a (2\$1s, 0)-meeting.

Proposition 1: The pure-strategy steady state exists for β and γ that are sufficiently close to one and zero, respectively, while the mixed-strategy steady state exists for β of intermediate size and γ sufficiently close to zero. Both steady states are locally stable and determinate. Moreover, the convergence to the pure-strategy steady state is gradual whereas that to the mixed-strategy steady state is by means of a jump in one period. Neither convergence involves the disposal of money.

In the pure-strategy steady state, \$2 bills do not circulate, so it serves as a model description of the circumstances that the largest bill (*e.g.*, the US \$100 bill) does not circulate. Our mixed-strategy steady state serves as an example in which small bills and large bills co-exist. While Lee *et al.* (2005) show the existence of a steady state in the general model, they do not tell what bills are actually circulating in that steady state. Our existence about the two steady states, though in a simpler model, suggests that discount factor β play a role in getting all denominations circulated.

IV. Nonfull-support steady states

Two steady states with non-full-support distribution exist. The equilibrium paths convergent to them will be the model description of public abandonment of the smaller denominations. To describe these non-full-support steady states, consider the following equation, which is equivalent to the last equation in (9) if we let $w_2=h_2=x$, $\pi_1=0$, and $\max\{u(\beta w_2), \beta w_2\}=u(\beta w_2)$. That is

$$x=\frac{n+m-1}{n}\beta x+\frac{1-m}{n}u(\beta x)-\gamma. \tag{11}$$

Figure 1 depicts the RHS of (11) of different values of γ. The highest curve corresponds to the case of $\gamma=0$. As long as $u'(0)$ is large enough to satisfy the condition for the existence of steady state in Trejos, and Wright (1995)

$$u'(0)>\frac{n(1-\beta)}{\beta(1-m)}+1,$$

this curve crosses the 45 degree line twice, and Equation (11) has two solutions, $\bar{w}>0$ and $\underline{w}=0$.

As γ increases, the curve is shifted downward, and the two solutions become positive and move toward each other (Figure 1). Eventually, these two solutions merge at some value w^*, when γ reaches some $\gamma^*>0$. In this process, $u(\beta\bar{w}-\beta\underline{w}\})$ decreases to zero, while $\beta\underline{w}$ increases from zero. Hence, there exists $\gamma'\in(0, \gamma^*)$ such that $u(\beta\bar{w}-\beta\underline{w})=\beta\underline{w}$.

Based on the tangency in Figure 1, w^* is the unique solution to

$$\frac{n+m-1}{n}\beta+\frac{1-m}{n}\ u'(\beta w^*)\beta=1. \tag{12}$$

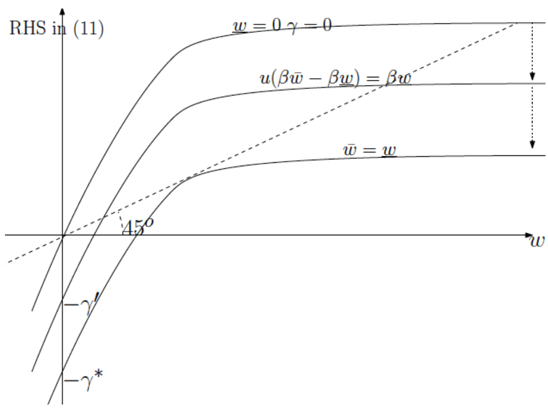

FIGURE 1
γ ON EQUATION (11)

By assuming that the curve in Figure 1 tangent to 45 degree line at w^* is sufficiently concave, we can make sure that $((1-m)/n)\beta w^*<\gamma^*$, which in turn implies Equation (19) when $\gamma=\gamma^*$. Letting $x=\underline{w}$ and rearranging the terms in Equation (11) implies the following:

$$\frac{(1-\beta)n+2-2m}{n}\underline{w}=\frac{1-m}{n}[u(\beta\underline{w})+\beta\underline{w}]-\gamma. \tag{13}$$

As γ increases in Figure 1, the LHS of Equation (19) increases as $\underline{w}$; hence, the LHS of Equation (13) increases, while the RHS of Equation (19) decreases as $\bar{w}$ increases. Overall, Equation (19) holds when $\gamma\in[0, \gamma^*]$. The following conditions are derived:

$$\text{(\$1, \$1)-meeting when } \gamma\in(\gamma', \gamma^*) \quad u(\beta\bar{w}-\beta\underline{w})<\beta\underline{w} \tag{14}$$

$$\text{(\$1, \$1)-meeting when } \gamma\in(0, \gamma') \quad u(\beta\bar{w}-\beta\underline{w})>\beta\underline{w} \tag{15}$$

$$\text{(\$1, 0)-meeting} \quad u(\beta\underline{w})>\beta\underline{w} \tag{16}$$

$$\text{(\$2, \$1)-meeting} \quad u(\beta\bar{w}-\beta\underline{w})>\beta\bar{w}-\beta\underline{w} \tag{17}$$

$$\text{(\$2, 0)-meeting} \quad u(\beta\bar{w})>\beta\bar{w} \tag{18}$$

Table 3

$\gamma \in (0, \gamma')$

Portfolio stage	Always choose \$2 bills if possible				
			Seller's wealth		
			0	1	2
Pairwise stage	Buyer's wealth	0	\	\	\
		1	\$1	\$1	\
		2	\$2	\$1	\

$$\text{\$2 } vs. \text{ 2\$1s} \qquad \frac{1-m}{n}[u(\beta \underline{w})+\beta \underline{w}]-\gamma<\frac{1-m}{n}u(\beta \bar{w}) \tag{19}$$

When $\gamma \in (0, \gamma')$, a nonfull-support steady state exists with $w_1 = h_1 = \underline{w}$, $w_2 = h_2 = \bar{w}$, $\pi = (1-m, 0, m)$ and the Table 3 strategy. h_{11} is defined by the second equation in (9). Equation (15)-(18)[2] are the optimal conditions in the corresponding meetings. Equation (19) is the optimal condition of choosing a \$2 over two \$1s.

When $\gamma \in (\gamma', \gamma^*)$, a steady state exists with the same formulas $w_1 = h_1 = \underline{w}$, $w_2 = h_2 = \bar{w}$, and $\pi = (1-m, 0, m)$. However, the optimal strategy is the Table 4 strategy, which is different from the above steady state. In particular, with Equation (14), no trade occurs in the (\$1, \$1)-meeting.

When $\gamma \in (0, \gamma^*)$, another non-full-support steady state exists with $w_1 = 0$, $h_1 = -\gamma$, $w_2 = h_2 = \bar{w}$, $\pi = (1-m, 0, m)$ and Table 5 strategy. h_{11} is defined by the second equation in (9). $w_1 = 0$ implies that one \$1 is not valued by a seller with 0 wealth. And meeting other potential sellers, namely those with one \$1, has zero probability. There is no benefit of carrying only one \$1, and it will be discarded to avoid carrying cost. An argument similar to the above implies the optimality of other parts of Table 5 strategy.

Then, stability about these steady states is considered. This paragraph involves two possible ways that the economy with initial distribution different from that of a steady state reaches the steady state. One is to jump to the steady state in one period, which possibly involves some people discarding some money at the initial date. The other convergent path involves people converting all \$1s into \$2 gradually.

Consider perturbing the distribution of Table 3 steady state while

[2] $u(\beta \bar{w}) > \beta \bar{w}$ because otherwise Equation (11) will be violated. Thus, Equation (16)-(18) hold.

keeping the total money stock constant. This is essentially a transfer payment that moves \$1 from the richest to the poorest in the model. Because it is strictly optimal to pay a \$1 in (\$1, \$1)-meetings and convert the two \$1s into one \$2 at the steady state, because it is strictly optimal to do so at the steady state, when the perturbation is sufficiently small. This process has a peculiar property that we call *unit-root convergence*.

That is, as π_1 approaches zero, the frequency of (\$1, \$1)-meetings reaches zero much faster; as a result, the convergence becomes extremely slow in the end. Nevertheless, such convergent path to the steady state exists. As proof, we derive a difference equation system of three variables (π_1^t, w_1^t, w_2^t) and study the three eigenvalues of its linearization around the steady state. The stable manifold is shown to be two-dimensional (2D).[3] Given that the initial condition is one-dimensional (1D), that is, π_1^0 only has one initial condition, a continuum of equilibrium paths converging to the steady state exists. This steady state is concluded to be locally stable and indeterminate.[4] Note that this local stability is not evident. When $\gamma=0$, such a convergent equilibrium does not exist. As the proposition of Huang, and Igarashi (2014) implies, the non-full-support steady state becomes unstable when $\gamma=0$.

Consider the above perturbation on the Table 4 steady state. Reserving a \$1 is strictly preferred in (\$1, \$1)-meetings at the steady state, and is thus also strictly preferred for a sufficiently small perturbation. Each \$1 never meets another \$1; hence, those \$1s cannot be converted into \$2s. The economy does not converge to this non-full-support steady state.

Consider injecting a sufficiently small proportion of \$1s into the Table 5 steady state. In other words, a positive proportion of agents with 0 wealth receive \$1s. Those receiving two \$1s will convert them to a \$2 bill immediately. Those with only one \$1 will discard it at the initial date to avoid carrying cost because discarding it is strictly optimal at the steady state; therefore, it is also sufficiently optimal near the steady state. The economy "jumps" in one period back to the non-full-support steady state with a lower stock of money because a proportion of \$1s is discarded.[5]

[3] Refer to the matrix computation in the Appendix.

[4] This indeterminacy has some resemblance to that of a non-monetary steady state of an overlapping generation model of fiat money with no carrying cost.

[5] This jump is possible even if $\gamma=0$. However, when $\gamma=0$, discarding money is a weakly dominated strategy. In addition, the jump by discarding when $\gamma=0$ will be eliminated by introducing even a tiny cost for discarding.

TABLE 4
$\gamma \in (\gamma', \gamma^*)$

Portfolio stage	Always choose \$2 bills if possible				
Pairwise stage			Seller's wealth		
			0	1	2
	Buyer's wealth	0	\	\	\
		1	\$1	0	\
		2	\$2	\$1	\

These results are summarized in the following proposition. The detailed proof is in Section V.

Proposition 2: Let $\gamma \in (0, \gamma^*)$. A non-full-support monetary steady state exists with $w_1 > 0$[6]; it is locally stable (gradual convergence) and indeterminate. If $\gamma \in (0, \gamma')$, and then it becomes unstable if $\gamma \in (\gamma', \gamma^*)$.

A non-full-support monetary steady state with $w_1 = 0$ exists; if a positive proportion of wealth-0 agents receive only one \$1, the economy can jump to it in one period by discarding all \$1s, making it locally stable and determinate.

V. Proof Outlines

Express π_0^t and π_1^t in terms of π_1^t using $\sum_{i=0}^{2} \pi_i^t = 1$ and $\sum_{i=0}^{2} i\pi_1^t = 2m$:

$$(\pi_0^t, \pi_2^t) = (1 - m - \frac{\pi_1^t}{2}, m - \frac{\pi_1^t}{2}) \quad \text{where } \pi_1^t \in \Pi \equiv [0, 2\min\{m, 1-m\}]. \tag{20}$$

Let κ^t be the probability of paying \$1 in the (\$1, \$1)-meetings, and η^t the probability that an agent with wealth 2 chooses two \$1 bills at the portfolio stage in period t. Then the law of motion is

$$\pi_1^{t+1} = \pi_1^t - \frac{2(\pi_1^t)^2}{n} \kappa^t + \frac{2}{n} (1 - m - \frac{\pi_1^t}{2})(m - \frac{\pi_1^t}{2}) \eta^t \tag{21}$$

Under the conjecture that (i) discarding of money does not occur, that (ii) \$1 is transferred in (\$1, 0)-meetings and in (\$2, \$1)-meetings,

[6] When $\gamma = \gamma'$, both Tables 3 and 4 can be the strategy.

and that (iii) a positive amount of money is transferred in ($2, 0)-meetings, the Bellman equation, defined on $\mathbb{W} \equiv \{(w_1, w_2) \mid 0 \leq w_1 \leq w_2\}$, is expressed as follows:

$$w_1^t = \frac{n-1+\pi_2^t}{n}\beta w_1^{t+1} + \frac{\pi_0^t}{n}u(\beta w_1^{t+1}) + \frac{\pi_1^t}{n}\max\{u(\beta w_2^{t+1} - \beta w_1^{t+1}),\ \beta w_1^{t+1}\} - \gamma \tag{22}$$

$$\begin{aligned} w_2^t = {} & \frac{n-1+\pi_2^t}{n}\beta w_2^{t+1} \\ & + \max\left\{\frac{\pi_0^t}{n}[u(\beta w_1^{t+1}) + \beta w_1^{t+1}] - \gamma, \frac{\pi_0^t}{n}u(\beta w_2^{t+1})\right\} \\ & + \frac{\pi_1^t}{n}u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1} - \gamma \end{aligned} \tag{23}$$

The max operator in Equation (23) compares the option of carrying two bills and paying only one $1 with that of carrying only one bill and paying a $2 in ($2, 0)-meetings.

Proof of Proposition 2: Consider $\gamma \in (0, \gamma')$, such that $\kappa^t = 1$ and $\eta^t = 0$ hold sufficiently near the steady state. The dynamics resemble that for the non-full-support steady state in the model without a denomination structure (*i.e.*, $\gamma = 0$). The Jacobian of the dynamical system is identical to that of Proposition 5 by Huang, and Igarashi (2015), except that $\gamma > 0$ leads to a different steady-state value w. Through a similar computation, out of the three eigenvalues of the Jacobian of Equation (21)-(23), one eigenvalue from Equation (21) is unity, another is smaller than 1, and the other is greater than 1. Equation (21) implies a unit root convergence (see Figure 2 in Huang, and Igarashi (2015)) and that the stable manifold is 2D.

When $\gamma \in (0, \gamma')$, the steady state w is an interior point of $\mathbb{W}$. Then the standard approach (*i.e.*, studying the dimension of the stable manifold) applies. The two-dimensional stable manifold and one initial condition imply local stability and indeterminacy.

The proof for the stability of the pure-strategy steady state is standard. We derive a difference equation system of three variables (π_1^t, w_1^t, w_2^t), obtain the three eigenvalues of the Jacobian at the steady state, and show that the stable manifold is 1D. (This implies a unique path because we have only one initial condition.) The mixed-strategy steady state shows

TABLE 5
$\gamma \in (0, \gamma^*)$

Portfolio stage	Always choose \$2 bills if possible				
			Seller's wealth		
			0	1	2
Pairwise stage	Buyer's wealth	0	\	\	\
		1	\$1or 0	\$1	\
		2	\$2	\$1	\

that if the initial distribution is sufficiently close to that of the steady state, the economy can jump to the steady state by choosing appropriate randomization in the initial portfolio stage. To ensure that this convergence by jump is the only equilibrium path to the mixed-strategy steady state, the possibility of gradual convergence is ruled out. Such a convergent path will have to preserve the indifference condition for the mixed strategy along the path. By applying z-transform to the dynamical system, we show that preserving the indifference is generically not possible.

Proof of Proposition 1: The existence of the steady states is shown by a guess and verify process. We impose the strategies in Tables 1 and 2 on Equations (21)-(23) and show that the resulting (π, w) is consistent with the optimality conditions for the strategies:

(\$1, \$1)-meeting $$u(\beta\bar{w}-\beta\underline{w})>\beta\underline{w} \tag{24}$$

(\$1, 0)-meeting $$u(\beta\underline{w})>\beta\underline{w} \tag{25}$$

(\$2, \$1)-meeting $$u(\beta\bar{w}-\beta\underline{w})>\beta\bar{w}-\beta\underline{w} \tag{26}$$

2\$1s or \$2 and paying it $$\frac{\pi_0}{n}[u(\beta w_1)+\beta w_1]-\gamma>\frac{\pi_0}{n}u(\beta w_2) \tag{27}$$

2\$1s or \$2 and keeping it $$\frac{\pi_0}{n}[u(\beta w_1)+\beta w_1]-\gamma>\frac{\pi_0}{n}\beta w_2 \tag{28}$$

Inequality Equation (27) means that choosing to carry two \$1s and offering only \$1 in (2\$1s, 0)-meetings is at least as good as choosing to carry one \$2 and offering \$2 in (\$2, 0)-meetings. Inequality Equation (28) means that choosing to carry two \$1s and offering only \$1 in (2\$1s, 0)-meetings is better than choosing to carry one \$2 and reserving \$2 in (\$2, 0)-meetings. Equation (27) must maintain equality or the mixed-strategy steady state and a strict inequality for the pure-strategy steady state. The proof of existence is similar to that in Huang, and Igarashi (2014). It consists of three lemmas provided in the Appendix. Strict inequalities are important for the following stability analysis because the analysis guarantees that the inequalities also hold in the vicinity of the steady state.

The stability analysis for the pure-strategy steady state is standard. The 3×3 Jacobian is derived for the difference equation system (21)-(23), which is evaluated at the steady state. Then, we show that one eigenvalue from Equation (21) is smaller than 1, and the other two eigenvalues are greater than 1, making the stable manifold 1D.[7] This result shows the local stability and determinacy of the pure-strategy full-support steady state.

The stability of the mixed-strategy full-support steady state is twofold.

For the model without denomination or carrying cost, Huang, and Igarashi (2014) first show that if the initial distribution is close to that of the mixed-strategy full-support steady state, then the economy can reach the steady state in one period by people coordinating in the initial randomization. The same logic applies to our setting with carrying cost. Here, we show that such "jump" is the unique convergent path by ruling out the possibility of a gradual convergence. For that purpose we show that when the distribution gradually goes to that of the steady state, Equation (27) generically does not hold with equality.

Let $\Delta\eta(z)$, $\Delta\pi_1(z)$, and $\Delta w(z)$ be the z-transforms[8] of $\{\Delta\eta^t\}_0^\infty$, $\{\Delta\pi_1^t\}_0^\infty$, and $\{\Delta w^t\}_0^\infty$, respectively, where the Δs in the latter indicate the deviation from the steady-state values. The RHS of the law of motion Equation (21) and Bellman Equations (22)-(23) are denoted by Φ and ϕ respectively, and Φ_η, Φ_π, ϕ_π, and ϕ_w denote their derivatives with respect to the subscript variable evaluated at the steady state. By linearizing the equality condition in Equation (27) and conditions Equations (21)-(23) with respect to (π_1^t, w^t, η^t) at the steady-state value (π, w, η) and applying the z-transforms, we obtain the following:

[7] Refer to the Appendix for the matrix computation.

[8] The z-transform of a sequence of numbers $\{y_t\}$ is $Y(z) = \sum_{t=0}^{\infty} y_t/z^t$. Refer to sections 8.2-8.4 in Luenberger (1979) for a detailed discussion.

$$[u'(\beta w_1)+1][\Delta w_1(z)-\Delta w_1^0]=u'(\beta w_2)[\Delta w_2(z)-\Delta w_2^0] \tag{29}$$

$$\begin{pmatrix}\Delta\pi_1(z)\\ \Delta w(z)\end{pmatrix}=[Iz-A^1]^{-1}\left[\begin{pmatrix}\Phi_\eta\\ 0\end{pmatrix}\Delta\eta(z)+\begin{pmatrix}\Delta\pi_1^0\\ \Delta w^0\end{pmatrix}z\right], \tag{30}$$

where A^1 is the Jacobian of Equation (48) with (π, w) being the mixed-strategy steady state and $\zeta=1$.

With Equation (29) and multiplying Equation (30) by $(0 \quad u'(\beta w_1)+1 \quad -u'(\beta w_2))$, we obtain

$$\begin{aligned}(0 \quad u'(\beta w_1)+1 \quad -u'(\beta w_2))\mathrm{adj}[Iz-A^1]&\left[\begin{pmatrix}\Phi_\eta\\ 0\end{pmatrix}\Delta\eta(z)+\begin{pmatrix}\Delta\pi_1^0\\ \Delta w^0\end{pmatrix}z\right]\\ &=(z-\Phi_\pi)\,|\,Iz-\phi_w^{-1})\,|\,\{[u'(\beta w_1)+1]\Delta w_1^0\\ &\quad -u'(\beta w_2)\Delta w_2^0\},\end{aligned} \tag{31}$$

where adj $[Iz-A^1]$ is the adjoint matrix of $Iz-A^1$. With the form of A^1, manipulating the algebra produces the first term in the LHS of (31) equal to $\Delta\eta(z)$ times $-(u'(\beta w_1)+1 \quad -u'(\beta w_2))$ adj $(z-\phi_w^{-1})\phi_w^{-1}\,\phi_\pi\Phi_\eta$, which has a nonzero constant generically because the shape of function u can be modified so that $u'(\beta w_1)$ and $u'(\beta w_2)$ can be arbitrarily changed without modifying $u(\beta w_1)$ and $u(\beta w_2)$. Equation (31) holds as an identity only if $\Delta\eta(z)=\Delta\eta^0$. For any z, Equation (30) holds; if we let $z=0$, the unique path has $\begin{pmatrix}\Delta\pi_1^0\\ \Delta w^0\end{pmatrix}=-(A^1)^{-1}\begin{pmatrix}\Phi_\eta\\ 0\end{pmatrix}\Delta\eta^0$, with $\Delta\pi_1^0$ given by the initial condition, and $\Delta\eta^t=\Delta\pi_1^t=\Delta w_1^t=\Delta w_2^t=0$ for all $t\geq 1$. This condition rules out a gradual convergence to the mixed-strategy steady state.

(Received 4 September 2014; Revised 2 October 2015; Accepted 10 November 2015)

Appendix

Supplementary materials for the detailed proofs (mainly for **Proposition 1**) are presented. Consider the following form of conditions Equation (27) and Equation (28):

Portfolio stage $$[u(\beta w_1)+\beta w_1]-\frac{n}{\pi_0}\gamma>u(\beta w_2) \tag{32}$$

2$1s *vs.* $2 $$[u(\beta w_1)+\beta w_1]-\frac{n}{\pi_0}\gamma>\beta w_2 \tag{33}$$

Lemma 1: *If a monetary full-support steady state exists for a sufficiently small $\gamma>0$, then:*

i. Equations (24)-(33) hold;
ii. π_1 satisfies $\pi_1 \le \pi_1^* \equiv (\sqrt[2]{1+12m(1-m)}-1)/3$, where inequality is strict if and only if $\eta^t<1$.

Proof. The proof studies the fixed point of Equations (21)-(23).

(i) Being a monetary steady state implies $w_2>0$, and having a full-support distribution implies $\pi_1>0$. Then, Equation (22) implies $w_1>0$ for a sufficiently small γ. We call the weak inequality versions of Equations (24)-(28), where the only change made in Equations (24)-(28) is replacing the strong inequalities by weak ones, and Equations (24)-(28) hold at least weakly.

Then, we obtain the following:

Step 1: Any full-support monetary steady state satisfies Equations (24)-(28) at least weakly for a sufficiently small γ.

Proof of Step 1

Recall that the probability of paying $1 in the ($1, $1)-meetings is denoted by κ in the main text. First, we want to show $\kappa>0$, and that Equation (24) and Equations (32)-(33) hold at least weakly.

Suppose by way of contradiction, both Equation (24) and Equation (25) hold with a reversed weak inequality. We can derive from Equation (22) that $w_1<0$, which contradicts $w_1>0$.

Suppose by contradiction that Equation (25) and the reversed strict inequality in Equation (24) hold. These imply the following:

$$\beta w_2-\beta w_1<\beta w_1 \tag{34}$$

Note that Equation (25) implies $0<\beta w_1<\bar{x}$, with $\bar{x}$ being the positive fixed point of $u(x)$. Thus, $0\le\beta w_2-\beta w_1<x$, which in turn implies Equation (26) at least weakly. This weak inequality and applying utility function $u(\cdot)$ on both sides of Equation (34) give Equation (33) for a sufficiently small γ. Given that u is strictly concave and Equation (24) does not hold, we have $u(\beta w_2)-u(\beta w_1)<\beta w_1$; hence, Equation (32) holds for a sufficiently small $\gamma>0$. Therefore, an agent with wealth 2 chooses two \$1s at the portfolio stage ($\eta=1$) for a sufficiently small γ. For π_1 to be strictly positive in Equation (21), we must have $\kappa>0$ and hence the weak Equation (24), a contradiction.

In summary, Equation (24) holds at least weakly.

Suppose by contradiction that either Equation (27) or Equation (28) holds with a reversed strict inequality or $\eta=0$ equivalently. Then, Equation (21) implies $\pi_1=0$, a contradiction to being a full-support steady state.

Suppose by contradiction that Equation (25) does not hold. Then, the weak Equation (24) implies $\beta w_2-\beta w_1>\beta w_1$. Combining this with the weak Equation (33) gives $u(\beta w_1)-(n\gamma/\pi_0)>\beta w_1$; hence, making Equation (25) for a sufficiently small γ a contradiction.

Suppose now by way of contradiction that Equation (26) does not hold even weakly: $u(\beta w_2-\beta w_1)<\beta w_2-\beta w_1$. Then, the weak Equation (33) for a sufficiently small γ implies $\beta w_2-\beta w_1<\beta w_1$. However, the weak Equation (24) and supposition imply $\beta w_2-\beta w_1>\beta w_1$, which is a contradiction. (*End of proof of Step 1*)

Step 2: If Equations (24)-(28) hold weakly, then Equations (24)-(26) and Equations (28) hold strictly.

Proof of Step 2

When Equations (24)-(28) hold at least weakly, "max" operators can be eliminated from Equations (22)-(23). Then, subtracting Equation (22) from Equation (23) presents the following:

$$w_2-w_2=\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}w_1-\frac{n}{n(1-\beta)+(1-\pi_2)\beta}\gamma, \tag{35}$$

and βw_1 satisfies

$$\frac{\beta}{n(1-\beta)+(1-\pi_2)\beta}[\pi_0\, u(\beta w_1)+ \\ \pi_1 u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1-\frac{n\beta}{n(1-\beta)+(1-\pi_2)\beta}\gamma\right)-n\gamma=\beta w_1. \tag{36}$$

Suppose by way of contradiction that Equation (24) does not hold. Then, we have

$$\beta w_1 \leq \frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\left(u(\beta w_1)-\frac{n\gamma}{\pi_0}\right) < u\left(\frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\beta w_1\right)$$

$$+\frac{n\gamma\beta}{n(1-\beta)+\pi_0\beta} < u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right)+\frac{n\gamma\beta}{n(1-\beta)+\pi_0\beta}$$

$$= u\left(\beta w_2 - \beta w_1 + \frac{n\beta}{n(1-\beta)+(1-\pi_2)\beta}\gamma\right)+\frac{n\gamma\beta}{n(1-\beta)+\pi_0\beta},$$

where the first inequality is obtained by substituting the supposition into Equation (36) and the second is by Jensen's inequality and strict concavity of u. For a sufficiently small γ, the above implies that Equation (24) holds.

Inequalities $u(\beta w_1) > u(\beta w_2 - \beta w_1) > \beta w_1 > \beta w_2 - \beta w_1$, where the first and third inequalities are by Equation (35) and the second is Equation (24), imply that inequality Equation (28) holds for a sufficiently small γ.

Suppose by way of contradiction that Equation (25) does not hold: $u(\beta w_1) \leq \beta w_1$. Then, Equation (24) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with Equation (33) produces $u(\beta w_1) > \beta w_1$, which is a contradiction.

Suppose now by way of contradiction that Equation (26) does not hold: $u(\beta w_2 - \beta w_1) \leq \beta w_2 - \beta w_1$. Then, Equation (33) implies $\beta w_2 - \beta w_1 \leq \beta w_1$. However, Equation (24) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction.

In summary, Equations (24)-(26) and Equation (28) hold strictly. (*End of proof of Step 2*)

(ii) Letting $\kappa = 1$ in Equation (21) and solving it for π_1 yield

$$\pi_1 = \sqrt{\left(\frac{\eta}{4-\eta}\right)^2 + 4m(1-m)\frac{\eta}{4-\eta}} - \frac{\eta}{4-\eta}.$$

Here, $\pi_1 \in [0, \pi_1^*]$ is strictly increasing in $\eta \in [0, 1]$ and is equal to π_1^* if $\eta = 1$. ■

Lemma 2: *A monetary full-support steady state exists for a sufficiently small γ if and only if there exists $(\pi_1, x) \gg 0$, such that*

$$x = \frac{\delta}{1-\pi_2}\left[\pi_0 u(x) + \pi_1 u\left(\delta x - \frac{\delta n\gamma}{(1-\pi_2)\beta}\right) - n\gamma\right] \equiv h(x, \pi_1, \gamma), \tag{37}$$

and

$$u\left[(1+\delta)x-\frac{\delta n\gamma}{(1-\pi_2)\beta}\right]\leq u(x)+x-\frac{n}{\pi_0}\gamma, \tag{38}$$

where

$$\delta=\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}<1, \tag{39}$$

and where Equation (38) must hold with equality if $\pi_1<\pi_1^*$.

Proof. (Necessity) Based on Lemma 1, inequalities Equations (24)-(28) hold for any full-support steady state if γ is sufficiently small. Under these inequalities, the Bellman Equations (22)-(23) become Equation (37) and Equation (39) with $x=\beta w_1$ and $(1+\delta)x-(\delta n\gamma/((1-\pi_2)\beta))=\beta w_2$. Additionally, Equation (32) implies Equation (38).

(Sufficiency) The proof resembles a guess-and-verify argument. Suppose we have (π_1, x), let $x=\beta w_1$ and $(1+\delta)x-(\delta n\gamma/((1-\pi_2)\beta))=\beta w_2$. Then, we have Equations (35)-(36). The same arguments as step 2 of the proof of Lemma 1 show that Equations (24)-(26) and Equation (28) hold, and Equation (27) is also given by Equation (38). Therefore, we have Equations (24)-(28), where Equation (27) holds with equality if and only if Equation (38) holds with equality. When $\pi_1=\pi_1^*$, with $\eta=1$ for the full-support steady states, the law of motion Equation (21) is satisfied at the steady state. When $\pi_1<\pi_1^*$, then Equation (38) holds with equality, and any η can be optimal. We choose the unique one that solves Equation (21). In summary, Lemma 1 trade is optimal.

Under such strategy, the Bellman Equation (22) and Equation (23) are equivalent to Equation (37) and Equation (39), respectively. ■

Lemma 3: *A monetary full-support steady state exists for a sufficiently small* $\gamma>0$ if and only if $u'(0)>(n(1-\beta))/(\beta(1-m))+1$.

Proof. In this proof, δ is denoted as $\delta(\pi_1)$ to make the dependence on π_1 explicit. First, we show the necessity that Lemma 2 implies Equation (37) with $x=\beta w_1$. Given that the RHS of Equation (37) is concave in x, to have a positive solution for a sufficiently small γ, we must have the following:

$$h_1\ (0, \pi_1, 0) = \delta_{\pi_1}\left[\frac{\pi_0}{1-\pi_2} + \frac{\pi_1\delta_{\pi_1}}{1-\pi_2}\right]u'(0) \equiv J_{\pi_1}u'(0) > 1, \tag{40}$$

or equivalently $u'(0) > (1/J_{\pi_1})$. Tedious algebra computation produces:

$$\begin{aligned}\frac{1}{J_{\pi_1}} &= \frac{n(1-\beta)}{\beta(1-m)} + 1 + \frac{n(1-\beta)}{\beta}\ \frac{\pi_1\, n(1-\beta) + \beta\pi_1\,\pi_0}{[\pi_0\, n(1-\beta) + (1-\pi_2)^2\,\beta](2-2m)} \\ &\geq \frac{n(1-\beta)}{\beta(1-m)} + 1,\end{aligned} \tag{41}$$

which completes the argument for necessity.

Next, we focus on sufficiency. Notice that Equation (41) holds with equality if and only if $\pi_1 = 0$. Then, the assumption implies that $h_1\ (0, 0, 0) = J_0\, u'\ (0) > 1$. Equivalently, the equation $h(x, 0, \gamma) = x$ has two solutions for any sufficiently small γ. $h_1\ (x_0, 0, \gamma) = \delta_0\, u'(x_0) < 1$, with x_0 being the larger solution, because $h(x, 0, \gamma)$ is concave in x.

When $\pi_1 = 0$, we have

$$\begin{aligned}&u\left[(1+\delta_0)x_0 - \frac{\delta_0 n\gamma}{(1-\pi_2)\beta}\right] - u(x_0) + \frac{n\gamma}{\pi_0} < u'(x_0)\delta_0 x_0 \\ &-\frac{u'(x_0)\delta_0\, n\gamma}{(1-\pi_2)\beta} + \frac{n\gamma}{\pi_0} < x_0 - \frac{n\gamma}{\pi_0\,\beta} + \frac{n\gamma}{\pi_0} < x_0,\end{aligned} \tag{42}$$

where the first inequality is by concavity of $u(\cdot)$, and the second is by $\delta_0\, u'(x_0) < 1$. Two cases are presented below.

Case 1: There exists $\bar{\pi}_1 \in (0, \pi_1^*)$, such that $h_1(0, \bar{\pi}_1, 0) = 1$ and $h_1(0, \pi_1, 0) > 1$ for all $\pi_1 \in (0, \bar{\pi}_1)$. Then, for any sufficiently small γ, Equation (37) has two positive solutions for all $\pi_1 \in (0, \bar{\pi}_1)$. The larger one x_{π_1} is considered a function of π_1.

Then, we have

$$\lim_{\pi_1 \to \bar{\pi}_1} \lim_{\gamma \to 0} \frac{u(x_{\pi_1}) + x_{\pi_1} - \frac{n\gamma}{\pi_0} - u\left[(1+\delta_{\pi_1})x_{\pi_1} - \frac{\delta_{\pi_1} n\gamma}{(1-\pi_2)\beta}\right]}{x_{\pi_1}} =$$

$$\lim_{\pi_1 \to \bar{\pi}_1} \frac{u(x_{\pi_1}) + x_{\pi_1} - u[(1+\delta_{\pi_1})x_{\pi_1}]}{x_{\pi_1}} < \lim_{\pi_1 \to \bar{\pi}_1} \frac{x_{\pi_1} - u'[(1+\delta_{\pi_1})x_{\pi_1}]\delta_{\pi_1} x_{\pi_1}}{x_{\pi_1}} \tag{43}$$

$$= 1 - u'(0)\delta_{\bar{\pi}_1} = \frac{\bar{\pi}_1(\delta_{\bar{\pi}_1} - 1)}{1 - \bar{\pi}_2} u'(0)\delta_{\bar{\pi}_1} < 0.$$

The first inequality follows from the concavity of u. The second equality uses the fact that $x_{\pi_1} \to 0$ as $\pi_1 \to \bar{\pi}_1$ and $\gamma \to 0$. The last equality uses $h_1(0, \bar{\pi}_1, 0) = 1$. The last inequality is by $\delta_{\bar{\pi}_1} < 1$.

With the limiting condition of Equation (43), a π_1 can be found sufficiently close to $\bar{\pi}_1$ such that Equation (38) holds with reversed strict inequality for any sufficiently small γ. As π_1 increases in $[0, \bar{\pi}_1]$, the inequality in Equation (38) switches directions, and the intermediate value theorem implies the existence of $\hat{\pi}_1 \in (0, \bar{\pi}_1)$ such that

$$u\left[(1 + \delta_{\hat{\pi}_1})x_{\hat{\pi}_1} - \frac{\delta_{\hat{\pi}_1} n\gamma}{(1 - \hat{\pi}_2)\beta}\right] = u(x_{\hat{\pi}_1}) + x_{\hat{\pi}_1} - \frac{n}{\hat{\pi}_0}\gamma, \tag{44}$$

Based on Lemma 2, such a pair $(\hat{\pi}_1, x_{\hat{\pi}_1})$ forms a mixed-strategy full-support steady state.

Case 2: $h_1(0, \pi_1, 0) > 1$ for all $\pi_1 \in [0, \pi_1^*]$. As in Case 1, view the larger solution to Equation (37), x_{π_1}, as a function of π_1. If Equation (38) holds with reversed inequality at π_1^*, then the intermediate value theorem and Lemma 2 imply the pair $(\hat{\pi}_1, x_{\hat{\pi}_1})$ forms a mixed-strategy full-support steady state; otherwise, Lemma 2 implies that there is a (pure-strategy) full-support steady state.

Lemma 1 rules out other possible full-support steady states. ■

The stability analysis on all steady states considers the following matrices:

$$\Phi_\pi^\zeta = 1 - \frac{\sqrt{1 + 12m(1-m)}}{n}\zeta, \tag{45}$$

$$\phi_\pi^\zeta = \begin{pmatrix} \dfrac{-\beta w_1 - u(\beta w_1) + 2u(\beta\Delta w)}{2n} \\ \dfrac{-\beta w_2 - [\zeta\{\beta w_1 + u(\beta w_1)\} + (1-\zeta)u(\beta w_2)] + 2[u(\beta\Delta w) + \beta w_1]}{2n} \end{pmatrix}, \tag{46}$$

$$\phi_w^\zeta = \begin{pmatrix} \dfrac{(n - 1 + \pi_2)\beta + \pi_0\beta u'(\beta w_1) - \pi_2\beta u'(\beta\Delta w)}{n} & \dfrac{\pi_2 u'(\beta\Delta w)}{n} \\ \dfrac{\zeta\pi_0\beta(u'(\beta w_1) + 1) + \pi_1\beta(1 - u'(\beta\Delta w))}{n} & \dfrac{(n - 1 + \pi_2)\beta + \pi_0(1-\zeta)\beta u'(\beta w_2) - \pi_2\beta u'(\beta\Delta w)}{n} \end{pmatrix}, \tag{47}$$

and

$$A^{\zeta} = \begin{pmatrix} \Phi_{\pi}^{\zeta} & 0 \\ -(\Phi_{w}^{\zeta})^{-1}\Phi_{w}^{\zeta} & (\Phi_{w}^{\zeta})^{-1} \end{pmatrix}, \tag{48}$$

where $\Delta w \equiv w_2 - w_1$ and $\zeta \in \{0, 1\}$.[9] The following contains the existence argument and the matrix computation in the Proof of Proposition 1.

Remaining Proof of Proposition 1: When β is sufficiently close to 1, the pure-strategy full-support steady state exists for any sufficiently small γ. To verify this, let $\pi_1 = \pi_1^*$, and take the limiting process of first $\gamma \to 0$; then, $\beta \to 1$. Conditions Equations (37), (38), and Equation (39) approach $u(x_{\pi_1^*}) = x_{\pi_1^*}$, $u(2x_{\pi_1^*}) = u(x_{\pi_1^*}) + x_{\pi_1^*}$, and $\delta_{\pi_1^*} = 1$, respectively. On the basis of the strict concavity of u, the limiting condition of (38) holds. Thus, the pure-strategy full-support steady state exists for (β, γ), which is sufficiently close to (1, 0).

The proof of Lemma 3 implies that a mixed-strategy full-support steady state exists for any sufficiently small γ if a pure-strategy state does not exist; and that $J_{\pi_1^*} u'(0) > 1$ is necessary for the existence of a pure-strategy full-support steady state. Based on Lemma 3, if $u'(0) \in ((n(1-\beta))/(\beta(1-m)), 1/J_{\pi_1^*})$,[10] then a mixed-strategy full-support steady state exists for any sufficiently small γ, which implies a generic existence.

Next, we turn to the stability of the pure-strategy full-support steady state. Trading one unit in all trade meetings is a strictly preferred strategy at the steady state; hence, it is also optimal in its neighborhood. The Jacobian of this steady state at the steady state by (π^*, w^*) is Equation (48) with $\zeta = 1$. Owing to the top-right submatrix being a zero matrix, one eigenvalue is given by Equation (45), which is smaller than 1, and the other two eigenvalues are the reciprocals of the eigenvalues of ϕ_w^{ζ}. In what follows, the eigenvalues of ϕ_w^{ζ} are shown to be smaller than 1 in absolute value.

Given that $h_1(\beta w_1, \pi_1^*, \gamma)$ is concave in w_1 and $h_1(\beta w_1^*, \pi_1^*, \gamma)$, we have

$$\frac{n(1-\beta) + (1-\pi_2^*)\beta}{\beta} > \pi_0^* u'(\beta w_0^*) + \pi_1^* \frac{(1-\pi_2^*)\beta}{n(1-\beta) + (1-\pi_2^*)\beta} u'(\beta \Delta w^*). \tag{49}$$

[9] ζ is just a parameter. $\zeta = 1$ and $\zeta = 0$ correspond to full-support steady states and non-full-support steady states respectively.

[10] Equation (41) implies that this set is non-empty. The condition is satisfied for β of intermediate value.

The eigenvalues of a general 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by

$$\eta_{+}, \eta_{-} = \frac{a + b \pm \sqrt{(a-b)^2 + 4bc}}{2}. \tag{50}$$

Both eigenvalues are real because

$$(a-b)^2 + 4bc = \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)\right]^2 + 4\ \frac{1-\pi_2^*}{n}\beta\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) > 0.$$

They are smaller than 1 in absolute value if and only if $a+d<2$ and $(1-a)(1-d)-bc>0$. An algebra computation produces

$$\begin{aligned} 1-a+1-d &= 2\left(1-\frac{n-1+\pi_2^*}{n}\beta\right) - \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) > \\ & 2\,\frac{n(1-\beta)+(1-\pi_2^*)\beta}{n} - \frac{\pi_0{}^*}{n}\beta u'(\beta w_1^*) \\ & -\frac{\pi_1{}^*}{n}\,\frac{(1-\pi_2{}^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}\,\beta u'(\beta\Delta w^*) > \frac{n(1-\beta)+(1-\pi_2^*)\beta}{n}; \end{aligned}$$

and

$$\begin{aligned} (1-a)(1-d)-bc &= \left(1-\frac{n-1+\pi_2^*}{n}\beta - \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) + \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right) \\ &\quad \left(1-\frac{n-1+\pi_2^*}{n}\beta - \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right) - \\ &\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\left[\frac{1-\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right] = \\ &\frac{n(1-\beta)+(1-\pi_2^*)\beta}{n^2}\beta\left(\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta} - \pi_0^*\,u'(\beta w_1^*) - \pi_1^*\,\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}\,u'(\beta\Delta w^*)\right) > 0, \end{aligned}$$

where the last inequalities of the above two conditions follow from Equation (49). In summary, the Jacobian only has one eigenvalue smaller than 1 in absolute value. The pure-strategy full-support steady state has a 1D stable manifold. With the convergent path restricted by one initial condition, this full-support steady state is locally stable and determinate. ■

Matrix Computation in Proposition 2: The Jacobian (48) with $\zeta=0$ reduces to

$$A^{\gamma}=\begin{pmatrix} 1 & 0 & 0 \\ -r/a' & 1/a' & 0 \\ -s/d' & 0 & 1/d' \end{pmatrix}, \tag{51}$$

where

$$r\equiv\frac{1}{n}u(\beta w_2),\ s\equiv\frac{1}{2n}[u(\beta w_2)-\beta w_2]>0,\ a'\equiv\frac{(n-1+m)\beta}{n}+\frac{1-m}{n}\beta u'(0),$$

and

$$d'\equiv\frac{(n-1+m)\beta}{n}+\frac{1-m}{n}\beta u'(\beta w_2).$$

Note that because w_2 is the larger positive solution to Equation (11), $a'>1$ and $d'\in(0,1)$ hold. The eigenvalues of Equation (51) are its diagonal elements.

The law of motion has a unit root convergence. The associated eigenvector, which constitutes a base for the eigenspace, has the form $\begin{pmatrix} 1 \\ \frac{-r}{a'-1} \\ \frac{s}{1-d'} \end{pmatrix}$. ■

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